DISPERSION OF NON-NEWTONIAN FLUIDS

by

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B. S., National Taiwan University, 1962

A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Chemical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1965

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I. INTRODUCTION

Considerable interest has been shown in the study of residence-times in chemical equipment. This is due to the fact that the residence-time distribution gives valuable information on the nature of the flow through a plant. The total quantity of material inside a working plant is often difficult to determine accurately, and a calculation of the mean residence-time is valuable in giving the material inventory of a plant.

Consider the case of chemical reactions which, if not zero order or autocatalytic, decrease in rate as time proceeds. It is possible to calculate the average yield of the reaction by means of the following equation (1)

$$\frac{\bar{C}}{C_0} = \int_0^\infty f(\theta)\underline{E}(\theta)d\theta \qquad (1\text{-}1)$$

where $\bar{C}$ is the mean concentration of reagent left in the effluent, C_0 is the concentration of reagent in the feed, $f(\theta)$ is the kinetic equation which gives C/C_0 as a function of time in a batch reaction, and $\underline{E}(\theta)$ is the residence-time distribution function. This expression is only correct for first order chemical reactions. However, the residence-time study provides valuable information for the explanation of reactor behavior even in cases where the reaction is not first order.

Another field of fluid flow which is increasing in importance industrially is the behavior of fluids which do not obey Newton's law of viscosity in motion. Because of this negative definition of non-Newtonian behavior, no single equation can describe exactly the shear-stress and shear-rate relationships of all such materials over all ranges of shear rates. Numerous empirical equations have been proposed to express the steady state relation between the shear-stress and shear-rate. Among these the Ostwald-de Waele model and the

Bingham plastic model are the most generally useful two-constant models, and the Ellis model is the simplest three-constant model.

Most of the investigations of fluid dispersion have been confined to Newtonian fluids. But fluid dispersion in the processing of non-Newtonian fluids is as important as it is in the processing of Newtonian fluids. This thesis will be concerned mainly with the development of mathematical models which characterize the dispersion of non-Newtonian fluids in flow systems.

A generalized mathematical expression for diffusion in a flow system at constant temperature and pressure is (2)

$$\frac{\partial C}{\partial t} + V\cdot\nabla C = D\nabla^2 C + R \qquad (1\text{-}2)$$

where D is assumed to be independent of C.

Suppose that a fluid flow through a cylindrical tube with the symmetrical concentration distribution about the central line of the tube. When there is no chemical reaction in the system, Equation (2) becomes

$$\frac{\partial C}{\partial t} = D\left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial x^2}\right) - V_x(r)\frac{\partial C}{\partial x} \qquad (1\text{-}3)$$

It would be difficult to find a complete solution of Equation (3) which gives the value of C for all values of r, x and t when the distribution C at time $t = 0$ is known. However, approximate solutions can be found which are valid in certain limiting conditions. When the effects of both cross-sectional and longitudinal diffusions are negligible, the steady-state velocity profile becomes the only factor governing the apparent fluid dispersion. The flow model corresponding to such a condition is usually called the convective model. When the variation of axial velocity with cross-sectional position and the cross-sectional material transport by molecular diffusion are assumed to be the dominant dispersion mechanisms, the solution of Equation (3) is called the dispersion model. Both models are treated in this thesis for various non-Newtonian velocity distribution functions, such as Bingham plastic, Ostwald-de Waele and Ellis models and various flow geometries, such as cylindrical tubes, slits and open channels.

The analysis of turbulent flow of non-Newtonian fluids through smooth round tubes was performed for the first time by Dodge and Metzner (3). The analysis permitted the prediction of non-Newtonian turbulent velocity profiles. The dispersion model of turbulent non-Newtonian flow is also discussed in this thesis.

II. LITERATURE SURVEY

Fluid is passed through process equipment so that it may be modified in one way or another. To predict the performance of equipment, we must know the complete flow pattern of the fluid within the vessel. Because of the practical difficulties connected with obtaining and interpreting such information, an alternate approach is used, which requires knowledge only of how long different elements of fluid remain in the vessel. Though this partial information is not sufficient to completely define the nonideal flow within the vessel, it is relatively simple to obtain. The field today can be divided into several areas. One of these is how residence-time distribution functions can be measured experimentally. Another area which has had considerable attention is the theoretical derivation of the relationships between effective axial dispersion coefficients and dispersion models. It is the purpose of this chapter to review the present situation in this field.

The convective model of dispersion for Newtonian fluids in laminar flow was proposed by many investigators (4, 5) to predict residence-time distribution functions for flow in circular conduits. The cummulative age distribution at the outlet of the system, which corresponds to the response to a step function input of a tracer without entrance effect being considered, is expressed as

$$\underline{F}(\theta) = \begin{cases} 1 - \dfrac{1}{(2\theta)^2} & \text{for } \theta \geq \frac{1}{2} \\ 0 & \text{for } \theta \leq \frac{1}{2} \end{cases} \tag{2-1}$$

where θ is the dimensionless time defined by $\theta = t\bar{V}_x/L$. The exit age distribution, which corresponds to the response to a Dirac delta function input of a tracer, is

$$\underline{E}(\theta) = \frac{1}{2\theta^3} \qquad \text{for } \theta \geq \frac{1}{2}$$
$$= 0 \qquad \text{for } \theta \leq \frac{1}{2} \qquad (2\text{-}2)$$

If, however, the entrance effect cannot be neglected, the initial distribution of the tracer is uniform spacewise along the tube. For this case the cummulative and exit age distributions were found to be

$$\underline{F}(\theta) = 1 - \frac{1}{2\theta} \qquad \text{for } \theta \geq \frac{1}{2}$$
$$= 0 \qquad \text{for } \theta \leq \frac{1}{2} \qquad (2\text{-}3)$$

$$\underline{E}(\theta) = \frac{1}{2\theta^2} \qquad \text{for } \theta \geq \frac{1}{2}$$
$$= 0 \qquad \text{for } \theta \leq \frac{1}{2} \qquad (2\text{-}4)$$

In order to give a more precise description of the dispersion characteristics of fluids, axial and radial diffusion and the velocity profile should be considered simultaneously. In a series of papers (6, 7, 8), Taylor has treated the dispersion of soluble matter in a solvent flowing through a circular tube. For a Newtonian fluid in laminar flow, the distribution of concentration, C, of the soluble material depends on the balance between convection along the tube due to variation in velocity over the cross section and on cross-sectional molecular diffusion. The corresponding partial differential equation is

$$D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r}\right) = \frac{\partial C}{\partial t} + V_m\left(1 - \frac{r^2}{R^2}\right)\frac{\partial C}{\partial x} \qquad (2\text{-}5)$$

Here, D, the coefficient of molecular diffusion, is assumed to be independent of C. In general, the transfer of C along the tube by molecular diffusion is small compared with that produced by convection. It is thus assumed that $\frac{\partial^2 C}{\partial x^2}$ is negligible compared with $\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r}$. The condition necessary for this to be true can be expressed as (8)

$$D >> \frac{R^2 V_x^2}{48D} \qquad (2\text{-}6)$$

The transport equation therefore takes the form

$$D\left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r}\right) = \frac{\partial C}{\partial t} + V_m\left(1 - \frac{r^2}{R^2}\right)\frac{\partial C}{\partial x} \qquad (2\text{-}7)$$

subject to the boundary conditions that the wall of the tube is impermeable and that the concentration distribution is symmetrical with respect to the center line of the tube, or

$$\frac{\partial C}{\partial r} = 0, \qquad \text{at} \quad r = R \quad \text{and} \quad r = 0 \qquad (2\text{-}8)$$

It would be difficult to find the complete solution of Equation (7); on the other hand, certain approximate solutions do exist which are valid under certain limiting conditions. One such approximate solution considers dispersion by convection alone, which applies to the case when both the cross-sectional diffusion and longitudinal diffusion are negligible, and the steady state velocity profile becomes the only factor governing the apparent fluid dispersion. Some calculated distributions of C along a tube have been given by Taylor, which can be reduced to the residence-time distribution functions given in Equations (1) through (4). Another approximate solution suggested by Taylor is that the time necessary for a radial variation in C to die down owing to radial diffusion is much shorter than the time necessary for an appreciable change in C to occur through longitudinal convection. This can be expressed by the condition (6)

$$\frac{L}{\bar{V}_x} >> \frac{2R^2}{(3.8)^2 D} \tag{2-9}$$

where $\bar{V}_x$ is the mean speed of flow and L is the longitudinal extent of the region in which $\partial C/\partial x$ is appreciable.

It is convenient in the present discussion to define concentration and velocity relative to axes which move with the mean flow. The velocity relative to these axes is

$$V_{x1} = 2\bar{V}_x(1 - \frac{r^2}{R^2}) - \bar{V}_x = \bar{V}_x(1 - \frac{2r^2}{R^2}) \tag{2-10}$$

Letting $\xi = r/R$, the equation of diffusion becomes

$$\frac{\partial^2 C}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial C}{\partial \xi} = \frac{R^2}{D}\frac{\partial C}{\partial t} + \frac{R^2\bar{V}_x}{D}(1 - 2\xi^2)\frac{\partial C}{\partial x_1} \tag{2-11}$$

where $x_1 = x - \bar{V}_x t$. For small radial variations in C, partial equilibrium may be assumed, that is, the rate of change with time is equated to zero. Since the mean velocity across planes for which x_1 is constant is zero, the transfer of C across such planes depends only on the radial variation of C. In this calculation, $\partial C/\partial x_1$ is taken to be independent of ξ. The small radial variation in C can therefore be calculated from the equation

$$\frac{\partial^2 C}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial C}{\partial \xi} = \frac{R^2\bar{V}_x}{D}(1 - 2\xi^2)\frac{\partial C}{\partial x_1} \tag{2-12}$$

Since $\partial C/\partial x_1$ is assumed to be independent of ξ, and if C_m is the mean concentration over a section, $\partial C/\partial x_1$ is indistinguishable from $\partial C_m/\partial x_1$. Therefore, Equation (12) can be solved to give

$$C = C_m + \frac{R^2\bar{V}_x}{4D}\frac{\partial C_m}{\partial x_1}(-\frac{1}{3} + \xi^2 - \frac{1}{2}\xi^4) \tag{2-13}$$

The rate at which C is transported across a section at x_1 is

$$Q = 2\pi R^2 \int_0^1 \bar{V}_x (1 - 2\xi^2) C \xi \, d\xi \qquad (2\text{-}14)$$

Inserting the value of C from Equation (13), Equation (14) is found to be

$$Q = -\pi R^2 \left(\frac{R^2 \bar{V}_x^2}{48D}\right) \frac{\partial C_m}{\partial x} \qquad (2\text{-}15)$$

Taylor has concluded that the combined effects of longitudinal convection and radial molecular diffusion are equivalent to the transfer of the solute across planes which move with the mean speed of the flow and the rate of this transfer is equal to that which diffusivity, K, would give to a stationary fluid, as shown by

$$K = \frac{R^2 \bar{V}_x^2}{48D} \qquad (2\text{-}16)$$

The analogous problem of dispersion in turbulent flow can be solved in the same manner (7). Taylor has assumed in his analysis that the universal velocity distribution in a pipe and Reynolds' analogy that the transfer of matter, heat, and momentum by turbulence are exactly analogous. The apparent diffusion coefficient K under this case has been found to be $10.1RV_*$ or $K = 7.14R\bar{V}_x\sqrt{f}$. Here R is the radius of the pipe, $\bar{V}_x$ is the mean flow velocity, f is the Fanning friction factor and V_* the friction velocity.

In a later paper (9) Aris has presented a new analysis in which the restrictions imposed by Taylor are removed. He obtained this by fixing attention on the movement of the center of gravity of the distribution of solute and the growth of its higher moments. It has been shown that the rate of growth of the variance is proportional to the sum of the molecular diffusion coefficient D, and the Taylor dispersion coefficient $KR^2\bar{V}_x^2/D$. The value of K for the Newtonian system has been given by Taylor to be 1/48 for laminar flow and 10.1 for turbulent flow (6, 7).

Aris (10) has also applied this method to the case in which the solute can pass into another fluid phase flowing through a tube. The apparent diffusion coefficient has been shown to be the sum of the molecular diffusion coefficient and the Taylor dispersion coefficients in the two phases, and a term due to the finite rate of partition between them. In another paper (11), Aris has treated the case of a viscous flow under a pulsating pressure gradient. It has been found that the Taylor diffusion coefficient contains terms proportional to the square of the amplitude of the pressure pulsations. But the coefficients of such terms rarely contribute more than a fraction of 1/128 to the total dispersion coefficient.

Taylor's treatment of turbulent flow is valid only for high Reynolds numbers, however, because the velocity profile used in the treatment is valid only when the laminar sublayer and transition layers are negligibly small. Tichacek and co-workers (12) have refined Taylor's method by including the effect of molecular diffusion and by introducing the experimental velocity profiles rather than a generalized profile. They have included a first order approximation to $\partial C/\partial x$ as a function of the profile. They have also shown that the effects of axial turbulent diffusion are negligible compared to the mixing caused by radial differences in the velocity. According to their analysis, $D_L/\bar{V}_x d$ is dependent of the Reynolds number, pipe roughness, and Schmidt number. They have reported that their theoretical data are applicable with less than 25% error as shown by comparison with Taylor's experimental data.

Giddings and Seager (13) have theoretically and experimentally investigated a method similar in operation to chromatography techniques for measuring a wide range of diffusion coefficients. Their experimental work deals with gaseous diffusion coefficients measured at various flow velocities, concentrations, etc. The final equation is

$$\frac{\bar{V}_x h}{2} = D + \frac{R^2 \bar{V}_x^2}{48D} \qquad (2\text{-}17)$$

where h is the height equivalent to a theoretical plate in a typical gas chromatograph and the right-hand side is the sum of the molecular diffusion coefficient and Taylor's dispersion coefficient.

Bailey and Gogarty (14) have performed experiments in which a dilute solution of potassium permanganate is displaced by water. They also developed a numerical method and obtained an accurate solution of the mixing equation when longitudinal diffusion is neglected. They have reported that good agreement does exist between the numerical solution and the given experimental results, that the mixing zone lengths resulting from their experiment are nearly the same as those predicted by Taylor for a narrow range of dimensionless time, $\theta = Dt/R^2$. However, experiments at various flow times with a fixed velocity have shown that experimental dispersion coefficients increased slowly with time.

Bournia and co-workers (15) have measured the longitudinal dispersion of a finite slug of gas at various velocities by using a gas (1,3-butadiene) that absorbs light in the ultra-violet region and passing the dispersed slug through a narrow beam of ultra-violet light of wave-length 250 mμ. The results indicate that Taylor's approximation does not apply for values of $\bar{V}_x$ below 2 cm/sec, and that the criterion $\bar{V}_x R/D >> 6.9$ is very important while the upper criterion $4L/R >> \bar{V}_x R/D$ is less important because Taylor's approximation appears to apply best at the higher velocities.

Farrell and Leonard (16) also derived a solution for the laminar flow problem when the axial molecular diffusion is neglected. Their solution involved a series of eigen functions of the Laplace transformation of the concentration. The moments of the concentration distribution can readily be evaluated by making use of a computer.

Taylor and Aris both have concluded that an effective axial dispersion coefficient K can also be used in turbulent flow. This coefficient has been found to be a function of the well-known Fanning friction factor. Bischoff and Levenspiel (17) have extended Aris' theory to include a linear rate process, and have used the result to construct comprehensive correlations of dispersion coefficients.

Hawthorn (18) has considered the temperature effect of viscosity on the dispersion coefficient and has found that it can be altered by a factor of two in laminar flow, but that there is little effect for fully developed turbulent flow.

The analysis of dispersion accompanying the flow of a non-Newtonian fluid has recently been considered by Fan and Hwang (19). They have derived convective models for Bingham plastic and Ostwald-de Waele fluids. The dispersion model for Ostwald-de Waele fluids flowing through a circular tube has also been derived for which Taylor's results can be considered as a special case of Fan and Hwang's expressions.

III. CONVECTIVE MODELS

A general discussion of residence-time distributions for some flow models has been made by Fan and Hwang (19). First of all, they have treated the convective model extensively for fluids which can be represented by the Ostwald-de Waele and Bingham plastic models. In this chapter, the similarity of residence-time distributions between fluid flow through slits (opening between two parallel plates) and fluid flow through open channels will be verified. The convective model for the laminar flow of the Bingham plastic fluid which is more general than that derived by Fan and Hwang (19) will also be derived. Constant molecular diffusion coefficients, steady and iso-thermal flow situations, and constant vessel geometries will be assumed in the present analysis.

FLOW THROUGH OPEN CHANNELS

1. Bingham Plastic Model

The steady-state rheological behavior of Bingham plastic fluids can be expressed as (2)

$$\overset{\rightrightarrows}{\tau} = -\left\{\mu_0 - \frac{\tau_0}{\sqrt{\frac{1}{2}(\overset{\rightrightarrows}{\Delta} : \overset{\rightrightarrows}{\Delta})}}\right\}\overset{\rightrightarrows}{\Delta} \qquad \text{for } \frac{1}{2}(\overset{\rightrightarrows}{\tau} : \overset{\rightrightarrows}{\tau}) \geq \tau_0^{\,2} \qquad (3\text{-}1)$$

$$\overset{\rightrightarrows}{\Delta} = 0 \qquad \text{for } \frac{1}{2}(\overset{\rightrightarrows}{\tau} : \overset{\rightrightarrows}{\tau}) < \tau_0^{\,2} \qquad (3\text{-}2)$$

where μ_0 and τ_0 are parameters which characterize a fluid.

Under conditions for which a one-dimensional rheological statement in rectangular coordinates is valid (see Fig. (1)), Equations (1) and (2) reduce to

$$\tau_{yx} = \tau_0 - \mu_0 \frac{dV_x}{dy} \qquad \text{for } \tau_{yx} \geq \tau_0 \qquad (3\text{-}3)$$

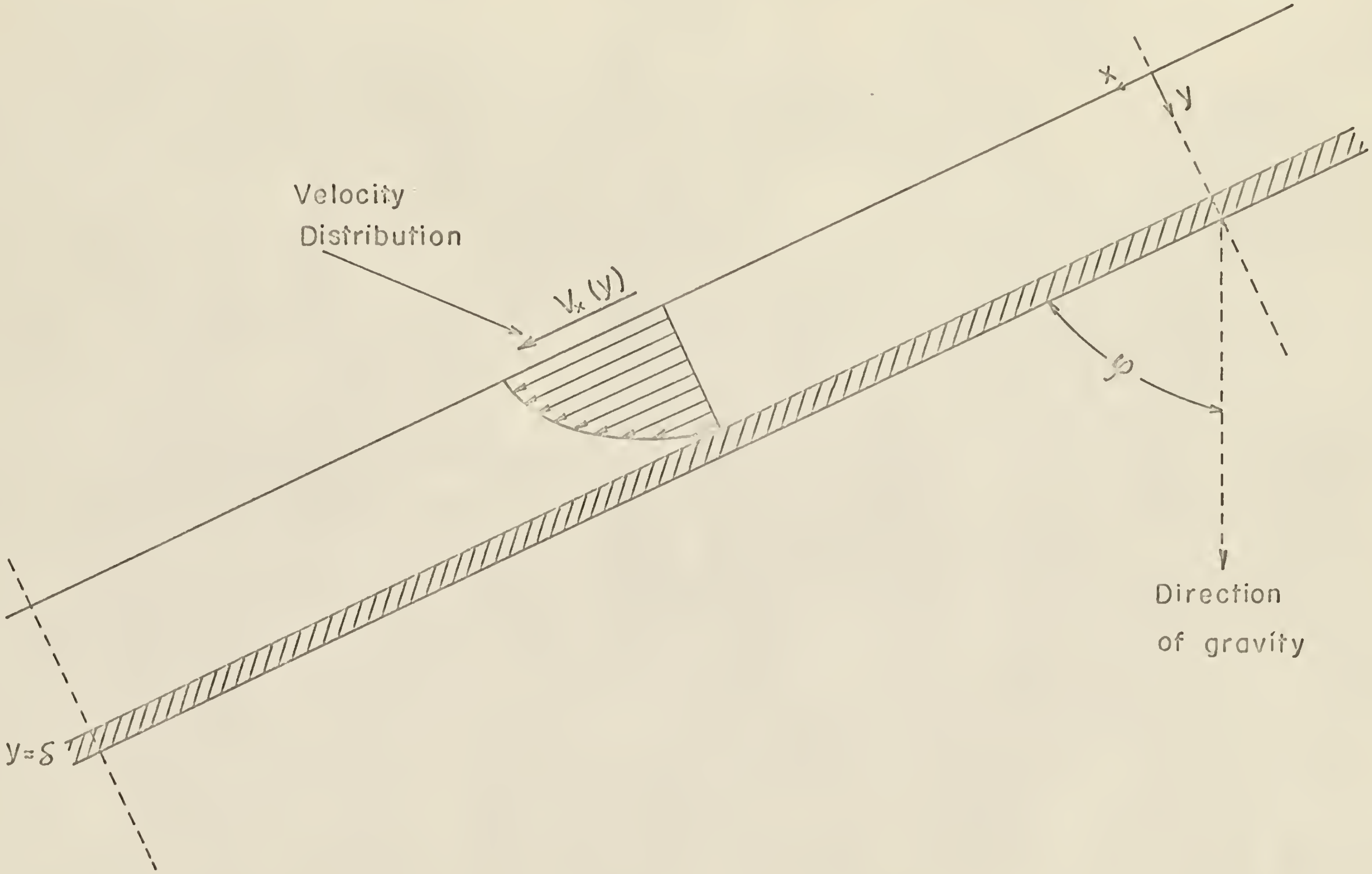

Fig. I. Flow of a viscous fluid film under the influence of gravity. The Z-axis is pointing outward from the plane of paper.

$$\frac{dV_x}{dy} = 0 \qquad \text{for} \quad \tau_{yx} < \tau_0 \tag{3-4}$$

respectively, and the equation of motion reduces to (2)

$$\frac{\partial}{\partial y}\tau_{yx} = \rho g \cos\varphi \tag{3-5}$$

Making use of the boundary conditions at the liquid-gas interface

$$\tau_{yx} = 0 \qquad \text{at} \quad y = 0 \tag{3-6}$$

The equation of motion may be integrated to give

$$\tau_{yx} = \rho g y \cos\varphi \tag{3-7}$$

Combining Equations (3) and (7), and integrating the resulting equation subject to the following boundary condition

$$V_x = 0 \qquad \text{at} \quad y = \delta \tag{3-8}$$

one obtains

$$V_x = -\frac{\rho g y^2 \cos\varphi}{2\mu_0} + \frac{\tau_0}{\mu_0} y + C_1 \tag{3-9}$$

and

$$C_1 = \frac{\rho g \delta^2 \cos\varphi}{2\mu_0} - \frac{\tau_0}{\mu_0}\delta \tag{3-10}$$

Hence, the velocity distribution is

$$V_x = \frac{\rho g \delta^2 \cos\varphi}{2\;{}_0}\left[1 - \left(\frac{y}{\delta}\right)^2\right] - \frac{\tau_0\delta}{\mu_0}\left(1 - \frac{y}{\delta}\right) \qquad \text{for } y \geq y_0 \tag{3-11}$$

$$V_m = \frac{\rho g \delta^2 \cos\varphi}{2\mu_0}\left[1 - \left(\frac{y_0}{\delta}\right)^2\right] - \frac{\tau_0\delta}{\mu_0}\left(1 - \frac{y_0}{\delta}\right) \qquad \text{for } y \leq y_0 \tag{3-12}$$

where

$$y_0 = \frac{\tau_0}{\rho g \cos\varphi}$$

Introducing the dimensionless distance variable $\zeta_0 = \frac{y_0}{\delta}$, the velocity distribution may also be written as

$$V_m = \frac{\rho g \delta^2 \cos\varphi}{2\mu_0} (1 - \zeta_0)^2 \qquad \text{for} \quad y \leq y_0 \qquad (3\text{-}13)$$

$$V_x = \frac{\rho g \delta^2 \cos\varphi}{2\mu_0} \left[(1 - \zeta_0)^2 - (\tfrac{y}{\delta} - \zeta_0)^2 \right]$$

$$= V_m \left[1 - \frac{(\frac{y}{\delta} - \zeta_0)^2}{(1 - \zeta_0)^2} \right] \qquad \text{for} \quad y \geq y_0 \qquad (3\text{-}14)$$

The mean velocity of the fluid is obtained by summing all the velocities over a cross-section and then dividing by the cross-sectional area, that is,

$$\bar{V}_x = \frac{\int_0^W \int_0^\delta V_x(y)\,dydz}{\int_0^W \int_0^\delta dydz} \qquad (3\text{-}15)$$

where W is the width of the channel. Therefore

$$\bar{V}_x = \frac{1}{\delta} \left\{ \int_0^{y_0} V_m \, dy + \int_{y_0}^{\delta} V_m \left[1 - \frac{(\frac{y}{\delta} - \zeta_0)^2}{(1 - \zeta_0)^2} \right] dy \right\}$$

$$= \frac{\rho g \delta^2 \cos\varphi}{2\mu_0} \left(\frac{2}{3} - \zeta_0 + \frac{1}{3} \zeta_0^3 \right) \qquad (3\text{-}16)$$

Dividing Equation (16) by Equation (13) yields

$$\bar{V}_x = \frac{2}{3} V_m \beta \qquad (3\text{-}17)$$

where

$$\beta = \frac{1 - \frac{3}{2}\zeta_0 + \frac{1}{2}\zeta_0^3}{(1 - \zeta_0)^2} = \frac{2 + \zeta_0}{2} \qquad (3\text{-}18)$$

When the entrance effect is negligible, the cumulative age distribution, which is a response to a unit step function input, is

$$\underline{F}(\theta) = \frac{1}{\delta W \bar{V}_x} \int_0^W \int_0^y V_x(y)dydz \qquad (3\text{-}19)$$

$$= \frac{1}{\delta \bar{V}_x} \int_0^{y_0} V_m \, dy + \frac{1}{\delta \bar{V}_x} \int_{y_0}^{y} V_m \left[1 - \frac{(\frac{y}{\delta} - \zeta_0)^2}{(1 - \zeta_0)^2} \right] dy \qquad (3\text{-}20)$$

$$= \underline{F}_1(\theta) + \underline{F}_2(\theta) \qquad (3\text{-}21)$$

where

$$\underline{F}_1(\theta) = \frac{1}{\delta \bar{V}_x} \int_0^{y_0} V_m \, dy = \frac{3\zeta_0}{2\beta} \cdot U_s \left(\theta - \frac{2}{3}\beta\right) \qquad (3\text{-}22)$$

From Equation (14) one has

$$\frac{y}{\delta} = \zeta_0 + (1 - \zeta_0) \sqrt{1 - \frac{V_x(y)}{V_m}}$$

$$= \zeta_0 + (1 - \zeta_0) \sqrt{1 - \frac{2\beta \, V_x(y)}{3\bar{V}_x}}$$

$$= \zeta_0 + (1 - \zeta_0) \sqrt{1 - \frac{2\beta}{3\theta}} \qquad (3\text{-}23)$$

and

$$\frac{d}{d\theta} \left(\frac{y}{\delta}\right) = \frac{\beta(1 - \zeta_0)}{3\theta^2 \sqrt{1 - \frac{2\beta}{3\theta}}} \qquad (3\text{-}24)$$

Substituting these equations into Equation (21) yields

$$\underline{F}_2(\theta) = \int_{\frac{2}{3}\beta}^{\theta} \frac{\beta(1 - \zeta_0)}{3\theta^3 \sqrt{1 - \frac{2\beta}{3\theta}}} \, d\theta$$

$$= (1 - \zeta_0)\left[\frac{\sqrt{\theta - \frac{2\beta}{3}}}{3\theta^{3/2}} + \frac{\sqrt{\theta - \frac{2\beta}{3}}}{\beta\theta^{1/2}}\right] \quad \text{for } \theta \geq \frac{2}{3} \qquad (3\text{-}25)$$

The response to a unit step function input is therefore

$$\underline{F}(\theta) = \frac{3\zeta_0}{2\beta} \cdot U_s(\theta - \tfrac{2}{3}\beta) + (1 - \zeta_0)\left[\frac{\sqrt{\theta - \frac{2\beta}{3}}}{3\theta^{3/2}} + \frac{\sqrt{\theta - \frac{2\beta}{3}}}{\beta\theta^{1/2}}\right]$$

$$\text{for } \theta \geq \frac{2}{3}\beta \qquad (3\text{-}26)$$

$$= 0 \qquad \text{for } \theta < \frac{2}{3}\beta$$

The $\underline{E}(\theta)$ function, which is the response to a Dirac delta function input of a tracer, is

$$\underline{E}(\theta) = \frac{3\zeta_0}{2\beta} \cdot \delta(\theta - \frac{2\beta}{3}) + \frac{(1 - \zeta_0)\beta}{3\theta^3} \cdot \frac{1}{\sqrt{1 - \frac{2\beta}{3\theta}}} \quad \text{for } \theta \geq \frac{2}{3}\beta$$

$$(3\text{-}27)$$

$$= 0 \qquad \text{for } \theta < \frac{2}{3}\beta$$

where $\delta(\theta - \frac{2}{3}\beta)$ is a Dirac delta function defined by

$$\delta(\theta - \tfrac{2}{3}\beta) = 0 \qquad \text{for } \theta \neq \tfrac{2}{3}\beta$$

$$= \infty \qquad \text{for } \theta = \tfrac{2}{3}\beta \qquad (3\text{-}28)$$

and

$$\int_{-\infty}^{\infty} \delta(\theta - \tfrac{2}{3}\beta)\, d\theta = 1 \qquad (3\text{-}29)$$

2. Ostwald-de Waele Model

The Ostwald-de Waele Model fluid may be characterized by the following relationship between the shear stress tensor and the rate of deformation tensor (2)

$$\vec{\vec{\tau}} = -\left\{ m \left| \sqrt{\frac{1}{2} (\vec{\vec{\Delta}} : \vec{\vec{\Delta}})} \right|^{\nu-1} \right\} \vec{\vec{\Delta}} \qquad (3\text{-}30)$$

Under conditions for which a one-dimensional rheological statement is valid, Equation (30) in rectangular coordinates may be written as (see Fig. (1))

$$\tau_{yx} = -m \left| \frac{dV_x}{dy} \right|^{\nu-1} \left(\frac{dV_x}{dy} \right) \qquad (3\text{-}31)$$

where m and ν are rheological parameters. Since $\frac{dV_x}{dy}$ is always negative for the given system, Equation (31) reduces to

$$\tau_{yx} = m \left(-\frac{dV_x}{dy} \right)^{\nu} \qquad (3\text{-}32)$$

The momentum-flux distribution has been found to be

$$\tau_{yx} = \rho g y \cos \varphi \qquad (3\text{-}7)$$

Substitution of Equation (32) into Equation (7) gives the following differential equation for the velocity distributions

$$-\frac{dV_x}{dy} = \left(\frac{\rho g y \cos \varphi}{m} \right)^{1/\nu} \qquad (3\text{-}33)$$

This equation is integrated subject to the boundary condition

$$V_x = 0 \qquad \text{at} \quad y = \delta \qquad (3\text{-}34)$$

to give

$$V_x = \left(\frac{\rho g \cos \varphi}{m} \right)^n \frac{\delta^{n+1}}{n+1} \left[1 - \left(\frac{y}{\delta} \right)^{n+1} \right] \qquad (3\text{-}35)$$

where $n = \frac{1}{\nu}$. The maximum velocity V_m occurs at $y = 0$; that is

$$V_m = \left(\frac{\rho g \cos \varphi}{m} \right)^n \frac{\delta^{n+1}}{n+1} \qquad (3\text{-}36)$$

Combining Equations (35) and (36) yields

$$V_x(y) = V_m \left[1 - \left(\frac{y}{\delta}\right)^{n+1} \right] \qquad (3\text{-}37)$$

The average velocity, $\bar{V}_x$, over a cross-section of the film is obtained as follows:

$$\bar{V}_x = \frac{1}{\delta} \int_0^{\delta} V_x(y)dy$$

$$= V_m \int_0^1 \left[1 - \left(\frac{y}{\delta}\right)^{n+1} \right] d\left(\frac{y}{\delta}\right)$$

$$= \frac{n+1}{n+2} V_m \qquad (3\text{-}38)$$

The cumulative age distribution of an Ostwald-de Waele fluid flowing through an open channel when the entrance effect is negligible is

$$\underline{F}(\theta) = \frac{1}{\delta \bar{V}_x} \int_0^{y} V_x(y)dy$$

$$= \frac{V_m}{\bar{V}_x} \int_0^{\frac{y}{\delta}} \left[1 - \left(\frac{y}{\delta}\right)^{n+1} \right] d\left(\frac{y}{\delta}\right) \qquad (3\text{-}39)$$

From Equation (37) one has

$$\frac{y}{\delta} = \left(1 - \frac{V_x}{V_m}\right)^{\frac{1}{n+1}}$$

$$= \left[1 - \frac{V_x(n+1)}{\bar{V}_x(n+2)} \right]^{\frac{1}{n+1}}$$

$$= \left[1 - \frac{n+1}{(n+2)\theta} \right]^{\frac{1}{n+1}} \qquad (3\text{-}40)$$

and

$$\frac{d}{d\theta}\left(\frac{y}{\delta}\right) = \frac{1}{(n+2)\theta^2}\left[1 - \frac{n+1}{(n+2)\theta}\right]^{\frac{-n}{n+1}} \qquad (3\text{-}41)$$

Substituting Equations (40) and (41) into Equation (39) shows that

$$\underline{F}(\theta) = \frac{1}{n+2}\int_{\frac{n+1}{n+2}}^{\theta} \frac{1}{\theta^3}\left[1 - \frac{n+1}{(n+2)\theta}\right]^{\frac{-n}{n+1}} d\theta \qquad (3\text{-}42)$$

Therefore

$$\begin{aligned} \underline{F}(\theta) &= \frac{1}{n+2}\int_{\frac{n+1}{n+2}}^{\theta} \frac{1}{\theta^3\left[1 - \frac{1}{\left(\frac{n+2}{n+1}\right)}\right]^{\frac{n}{n+1}}} d\theta \qquad \text{for } \theta \geq \frac{n+1}{n+2} \\ &= 0 \qquad \text{for } \theta < \frac{n+1}{n+2} \end{aligned} \qquad (3\text{-}43)$$

The residence-time distribution, which is the response to a Dirac delta function input, is obtained by taking the derivative of $\underline{F}(\theta)$ with respect to θ, that is,

$$\begin{aligned} \underline{E}(\theta) &= \frac{1}{n+2}\,\frac{1}{\theta^3\left[1 - \frac{1}{\left(\frac{n+2}{n+1}\right)\theta}\right]^{\frac{n}{n+1}}} \qquad \text{for } \theta \geq \frac{n+1}{n+2} \\ &= 0 \qquad \text{for } \theta < \frac{n+1}{n+2} \end{aligned} \qquad (3\text{-}44)$$

When the entrance effect is considered, the tracer is uniformly distributed in the entrance section. The cumulative age distribution at the outlet of the system, which corresponds to the response to a step function input of a tracer, is

$$\underline{F}(\theta) = \frac{1}{\delta}\int_0^y dy$$

Therefore

$$\underline{F}(\theta) = \int_{\frac{n+1}{n+2}}^{\theta} \frac{1}{(n+2)\theta^2} \cdot \frac{1}{\left[1 - \frac{1}{(\frac{n+2}{n+1})\theta}\right]^{\frac{n}{n+1}}} \, d\theta \,, \quad \text{for } \theta \geq \frac{n+1}{n+2}$$

$$= 0 \qquad \text{for } \theta < \frac{n+1}{n+2}$$

The response to a Dirac delta function input is then

$$\underline{E}(\theta) = \frac{1}{n+2} \cdot \frac{1}{\theta^2 \left[1 - \frac{1}{(\frac{n+2}{n+1})\theta}\right]^{\frac{n}{n+1}}} \qquad \text{for } \theta \geq \frac{n+1}{n+2}$$

$$= 0 \qquad \text{for } \theta < \frac{n+1}{n+2}$$

Comparing the distribution functions derived in this chapter for fluids flowing through an open channel with those derived by Fan and Hwang (19) for fluids flowing through a slit, it is seen that the forms of the velocity and concentration distributions are exactly identical. The differences arise from the fact that extensive properties, such as cross-sectional area, flow rate, and total tracer injected in this analysis are reduced to one half of those in Fan and Hwang's analysis. Thus, it is not surprising to find that both flow geometries result in the same residence-time distribution functions.

FLOW OF THE BINGHAM PLASTIC WITH SLIP VELOCITY AT THE TUBE WALL

The convective model for Bingham plastics will be treated by using a generalized velocity profile for which the slippage at the tube wall is considered. The momentum flux distribution for the flow of fluid through a circular tube has been given as (2) (see Fig. (2))

$$\tau_{rx} = \left(-\frac{P_0 - P_L}{2L}\right) r \qquad (3\text{-}45)$$

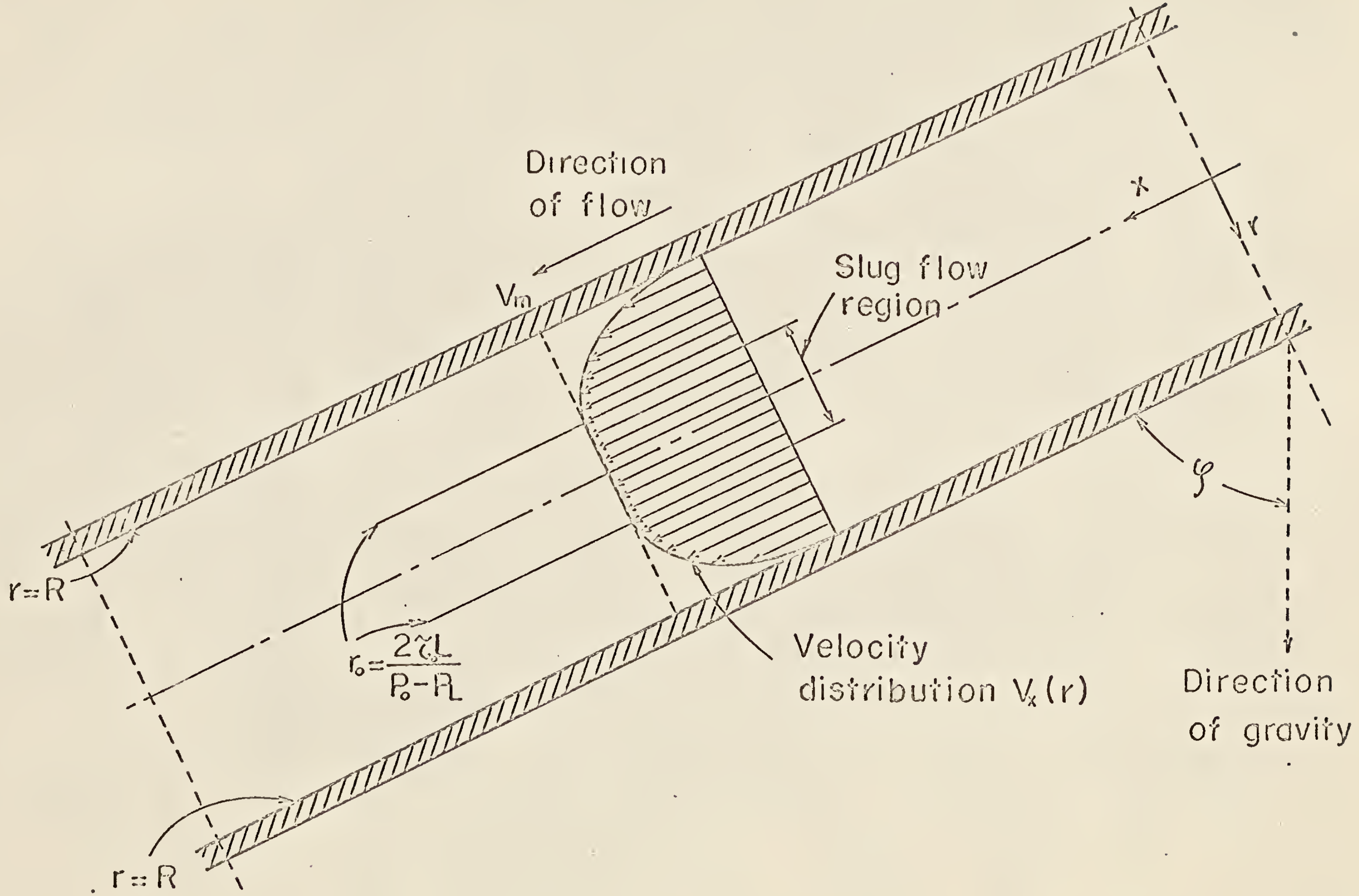

Fig. 2. Flow of a Bingham plastic in a circular tube.

in which $P = p - \rho gz$ and where p is the pressure acting on the given system.

The one-dimensional rheological statement of the Bingham plastic in cylindrical coordinates is found to be (2)

$$\tau_{rx} = \tau_0 - \mu_0 \frac{dV_x}{dr} \qquad \text{for } \tau_{rx} \geq \tau_0 \qquad (3\text{-}46)$$

$$\frac{dV_x}{dr} = 0 \qquad \text{for } \tau_{rx} < \tau_0 \qquad (3\text{-}47)$$

Combining Equations (45) and (46) and applying the boundary condition $V_x = V_R$ at $r = R$, the velocity distribution is

$$V_x(r) = V_R + (V_m - V_R)\left[1 - \frac{(\frac{r}{R} - \xi_0)^2}{(1 - \xi_0)^2}\right] \qquad (3\text{-}48)$$

where

$$\xi_0 = \frac{r_0}{R} = \frac{\tau_0}{\tau_R}$$

and V_m is the maximum velocity of the fluid. The average velocity of the fluid is calculated by summing all the velocities over a cross-section and then dividing by the cross-sectional area as

$$\bar{V}_x = \frac{\int_0^2 \int_0^R V_x(r) r dr d\theta}{\int_0^2 \int_0^R r dr d\theta} \qquad (3\text{-}49)$$

$$= 2\int_0^1 \left\{ V_R + (V_m - V_R)\left[1 - \frac{(\frac{r}{R} - \xi_0)^2}{(1 - \xi_0)^2}\right]\right\} (\tfrac{r}{R})\, d(\tfrac{r}{R})$$

$$= V_R + \frac{V_m - V_R}{2}\left[\frac{1 - \frac{4}{3}\xi_0 + \frac{1}{3}\xi_0^4}{1 - 2\xi_0 + \xi_0^2}\right]$$

$$= \frac{V_m}{2}\alpha + (1 - \frac{\alpha}{2})\, V_R \tag{3-50}$$

where

$$\alpha = \frac{1 - \frac{4}{3}\xi_0 + \frac{1}{3}\xi_0^{\,4}}{1 - 2\xi_0 + \xi_0^{\,2}} = \frac{1}{3}(3 + 2\xi_0 + \xi_0^{\,2}) \tag{3-51}$$

The minimum residence time of the fluid will be defined as

$$\theta_{\,min} = \frac{\bar{V}_x}{V_m} = \frac{\frac{V_m}{3}\alpha + (1 - \frac{\alpha}{3})V_R}{V_m}$$

$$= \frac{\alpha}{2\left[1 - (1 - \frac{\alpha}{2})\gamma\right]} \tag{3-52}$$

where γ is a slip velocity factor and is defined by $\gamma = V_R/\bar{V}_x$. An explicit expression of $\frac{r}{R}$ in terms of V_x can be obtained from Equation (48) as

$$\frac{r}{R} = \xi_0 + (1 - \xi_0)\sqrt{\frac{V_m - V_x}{V_m - V_R}} \tag{3-53}$$

Substituting Equation (52), $\gamma = V_R/\bar{V}_x$ and $\bar{V}_x/V_x = \theta$ into Equation (53) yields

$$\frac{r}{R} = \xi_0 + (1 - \xi_0)\sqrt{\frac{2\left[1 - (1 - \frac{\alpha}{2})\gamma\right] - \alpha/\theta}{2(1 - \gamma)}} \tag{3-54}$$

and

$$\frac{d}{d\theta}\left(\frac{r}{R}\right) = \frac{(1 - \xi_0)\alpha}{4\theta^2\sqrt{1 - \gamma}\,\sqrt{\left[1 - (1 - \frac{\alpha}{2})\gamma\right] - \alpha/2\theta}} \tag{3-55}$$

In response to a unit step function input, the cumulative age distribution is

$$\underline{F}(\theta) = \frac{1}{\pi R^2 \bar{V}_x} \int_0^{2\pi} \int_0^{r} V_x(r) r dr d\phi \qquad (3\text{-}56)$$

$$= \frac{2}{R^2 \bar{V}_x} \left[\int_0^{r_0} V_m \, rdr + \int_{r_0}^{r} \left\{ V_R + (V_m - V_R) \left[1 - \frac{(\frac{r}{R} - \xi_0)^2}{(1 - \xi_0)^2} \right] \right\} rdr \right] \qquad (3\text{-}57)$$

$$= \underline{F}_1(\theta) + \underline{F}_2(\theta) \qquad (3\text{-}58)$$

where

$$\underline{F}_1(\theta) = \frac{2}{R^2 \bar{V}_x} \int_0^{r_0} V_m \, rdr$$

$$= \frac{2\xi_0^{\,2}}{\alpha} \left[1 - (1 - \tfrac{\alpha}{2})\gamma \right] \cdot U_s \left\{ \theta - \frac{\alpha}{2\left[1 - (1 - \frac{\alpha}{2})\gamma\right]} \right\} \qquad (3\text{-}59)$$

and

$$\underline{F}_2(\theta) = \frac{2}{R^2 \bar{V}_x} \int_{r_0}^{r} \left\{ V_R + (V_m - V_R) \left[1 - \frac{(\frac{r}{R} - \xi_0)^2}{(1 - \xi_0)^2} \right] \right\} rdr$$

$$= \frac{2}{R^2} \int_{r_0}^{r} \left\{ \gamma + \left(\frac{2\left[1 - (1 - \frac{\alpha}{2})\gamma\right]}{\alpha} - \gamma \right) \left[1 - \frac{(\frac{r}{R} - \xi_0)^2}{(1 - \xi_0)^2} \right] \right\} rdr \qquad (3\text{-}60)$$

Substitution of Equations (54) and (55) into Equation (60) reveals that

$$\underline{F}_2(\theta) = \int_{\frac{\alpha}{2\left[1 - (1 - \frac{\alpha}{2})\gamma\right]}}^{\theta} \left[\frac{\xi_0}{\sqrt{1 - (1 - \frac{\alpha}{2})\gamma - \frac{\alpha}{2\theta}}} + \frac{1 - \xi_0}{\sqrt{1 - \gamma}} \right] \frac{(1 - \xi_0)\alpha}{2\sqrt{1 - \gamma}\,\theta^3} \, d\theta$$

$$= \frac{(1-\xi_0)\alpha}{\sqrt{1-\gamma}} \left\{ \frac{(1-\xi_0)\left[1-(1-\frac{\alpha}{2})\gamma\right]^2}{\sqrt{1-\gamma}\,\alpha^2} - \frac{1}{4\sqrt{1-\gamma}\,\theta^2} + \frac{\sqrt{2}\,\xi_0}{3\alpha} \, \frac{\sqrt{2\left[1-(1-\frac{\alpha}{2})\gamma\right]\theta-\alpha}}{\theta^{3/2}} \right.$$

$$\left. \frac{4\sqrt{2}\left[1-(1-\frac{\alpha}{2})\gamma\right]\xi_0}{3\alpha^2} \cdot \frac{\sqrt{2\left[1-(1-\frac{\alpha}{2})\gamma\right]\theta-\alpha}}{\theta^{1/2}} \right\} ,$$

$$\text{for} \quad \frac{\alpha}{2\left[1-(1-\frac{\alpha}{2})\gamma\right]} \leq \theta < \frac{1}{\gamma} \qquad (3\text{-}61)$$

The complete expression of $\underline{F}(\theta)$ is then

$$\underline{F}(\theta) = \frac{2\xi_0{}^2\lambda}{\alpha} U_s(\theta - \frac{\alpha}{2\lambda}) + \frac{(1-\xi_0)\alpha}{\sqrt{1-\gamma}} \left\{ \frac{(1-\xi_0)\lambda^2}{\sqrt{1-\gamma}\,\alpha^2} - \frac{1-\xi_0}{4\sqrt{1-\gamma}\,\theta^2} \right.$$

$$\left. + \frac{\sqrt{2}\,\xi_0}{3\alpha} \, \frac{\sqrt{2\lambda\theta-\alpha}}{\theta^{3/2}} + \frac{4\sqrt{2}\,\lambda\xi_0}{3\alpha^2} \, \frac{\sqrt{2\lambda\theta-\alpha}}{\theta^{1/2}} \right\}$$

$$\text{for} \quad \frac{\alpha}{2\lambda} \leq \theta \leq \frac{1}{\gamma} \qquad (3\text{-}62)$$

$$= 0 \qquad \text{for} \quad \theta \leq \frac{\alpha}{2\lambda}$$

$$= 1 \qquad \text{for} \quad \theta > \frac{1}{\gamma} .$$

where

$$\lambda = 1 - (1 - \frac{\alpha}{2})\gamma \qquad (3\text{-}63)$$

The residence-time distribution, which is the response to a Dirac delta function input, is

$$\underline{E}(\theta) = \frac{2\xi_0{}^2\lambda}{\alpha} \delta(\theta - \frac{\alpha}{2\lambda}) + \frac{(1-\xi_0)\alpha}{2\sqrt{1-\gamma}\,\theta^3} \left[\frac{\xi_0}{\sqrt{\lambda - \frac{\alpha}{2\theta}}} + \frac{1-\xi_0}{\sqrt{1-\gamma}} \right]$$

$$\text{for } \frac{\alpha}{2\lambda} \le \theta \le \frac{1}{\gamma} \qquad (3\text{-}64)$$

$$= 0 \qquad \text{elsewhere}$$

When the entrance effect cannot be neglected, the F-function can be calculated by

$$\underline{F}(\theta) = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^{r} r dr d\phi \qquad (3\text{-}65)$$

Thus

$$\underline{F}(\theta) = \frac{2\xi_0{}^2\lambda}{\alpha} \cdot U_s(\theta - \frac{\alpha}{2\lambda}) + \frac{(1-\xi_0)\alpha}{\sqrt{1-\gamma}} \left[\frac{2\xi_0\sqrt{\lambda - \frac{\alpha}{2\theta}}}{\alpha} - \frac{1-\xi_0}{2\theta\sqrt{1-\gamma}} + \frac{\lambda(1-\xi_0)}{\alpha\sqrt{1-\gamma}} \right]$$

$$\text{for } \frac{\alpha}{2\lambda} \le \theta \le \frac{1}{\gamma}$$

$$= 0 \qquad \text{for } \theta \le \frac{\alpha}{2\lambda} \qquad (3\text{-}66)$$

$$= 1 \qquad \text{for } \theta > \frac{1}{\gamma}$$

and the corresponding $\underline{E}$-curve is

$$\underline{E}(\theta) = \frac{2\xi_0{}^2\lambda}{\alpha} \delta(\theta - \frac{\alpha}{2\lambda}) + \frac{(1-\xi_0)\alpha}{2\sqrt{1-\gamma}\,\theta^2} \left[\frac{\xi_0}{\sqrt{\lambda - \frac{\alpha}{2\theta}}} + \frac{1-\xi_0}{\sqrt{1-\gamma}} \right]$$

$$\text{for } \frac{\alpha}{2\lambda} \le \theta \le \frac{1}{\gamma} \qquad (3\text{-}67)$$

$$= 0 \qquad \text{elsewhere}$$

When $\gamma = 0$, the slip velocity at the wall of the tube is zero, Equations (62), (64), (66) and (67) respectively become

$$\underline{F}(\theta) = \frac{2\xi_0{}^2}{\alpha} U_s(\theta - \frac{\alpha}{2}) + (1-\xi_0)\alpha \left[\frac{1-\xi_0}{\alpha^2} - \frac{1-\xi_0}{4\theta^2} \right.$$

$$\left. + \frac{\sqrt{2}\xi_0}{3\alpha} \frac{\sqrt{2\theta-\alpha}}{\theta^{3/2}} + \frac{4\sqrt{2}\,\xi_0}{3\alpha^2} \frac{\sqrt{2\theta-\alpha}}{\theta^{1/2}} \right]$$

$$\text{for} \quad \theta \geq \frac{\alpha}{2} \qquad (3\text{-}68)$$

$$= 0 \qquad \text{for} \quad \theta < \frac{\alpha}{2}$$

$$\underline{E}(\theta) = \frac{2\xi_0^{\,2}}{\alpha}\,\delta\left(\theta - \frac{\alpha}{2}\right) + \frac{(1-\xi_0)\alpha}{2\theta^3}\cdot\left(\frac{\xi_0}{\sqrt{1-\frac{\alpha}{2\theta}}} + 1 - \xi_0\right)$$

$$\text{for} \quad \theta \geq \frac{\alpha}{2} \qquad (3\text{-}69)$$

$$= 0 \qquad \text{for} \quad \theta < \frac{\alpha}{2}$$

$$\underline{F}(\theta) = \frac{2\xi_0^{\,2}}{\alpha}\,U_s\left(\theta - \frac{\alpha}{2}\right) + (1-\xi_0)\alpha\left[\frac{2\xi_0\sqrt{1-\frac{\alpha}{2\theta}}}{\alpha} - \frac{1-\xi_0}{2\theta} + \frac{1-\xi_0}{\alpha}\right]$$

$$\text{for} \quad \theta \geq \frac{\alpha}{2} \qquad (3\text{-}70)$$

$$= 0 \qquad \text{for} \quad \theta < \frac{\alpha}{2}$$

$$\underline{E}(\theta) = \frac{2\xi_0^{\,2}}{\alpha}\,\delta\left(\theta - \frac{\alpha}{2}\right) + \frac{(1-\xi_0)\alpha}{2\theta^2}\left(\frac{\xi_0}{\sqrt{1-\frac{\alpha}{2\theta}}} + 1 - \xi_0\right)$$

$$\text{for} \quad \theta \geq \frac{\alpha}{2} \qquad (3\text{-}71)$$

$$= 0 \qquad \text{for} \quad \theta < \frac{\alpha}{2}$$

When $\gamma = 0$ and $\xi_0 = 0$, the above distributions reduce to Equations (2-1), (2-2), (2-3), and (2-4) respectively, which have been derived for the Newtonian flow (4,5).

A family of numerically computed $\underline{F}$ and $\underline{E}$ curves with flow behavior index ξ_0 and slip velocity factor γ as parameters are shown in Figures (3) through (12). Since molecular diffusion has been neglected, certain points of discontinuity may occur in the distribution curves due to the irregular

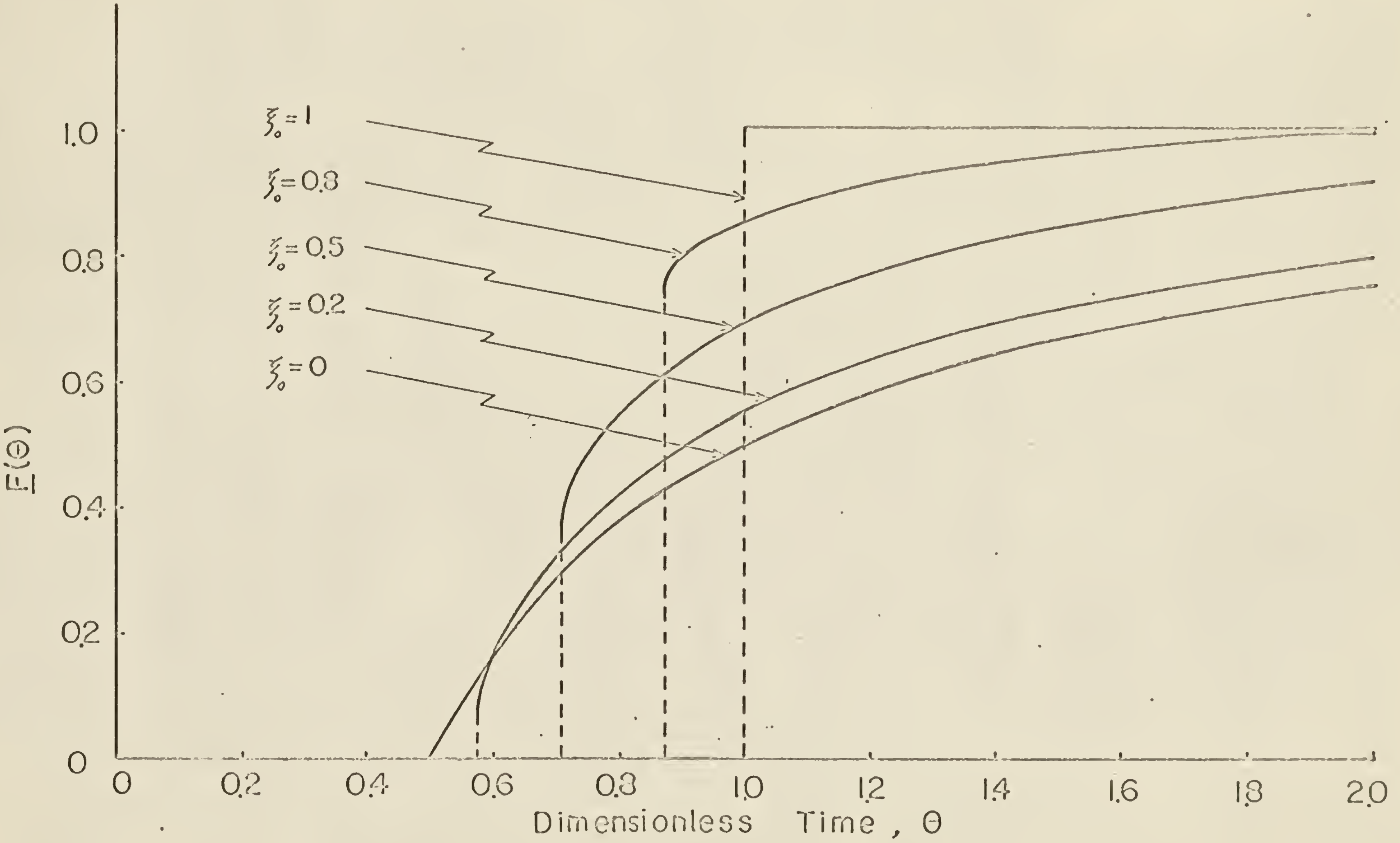

Fig. 3. $\underline{F}$ curves for laminar flow of Bingham model fluids in a cylindrical tube with entrance effect considered and $\eta = 0$.

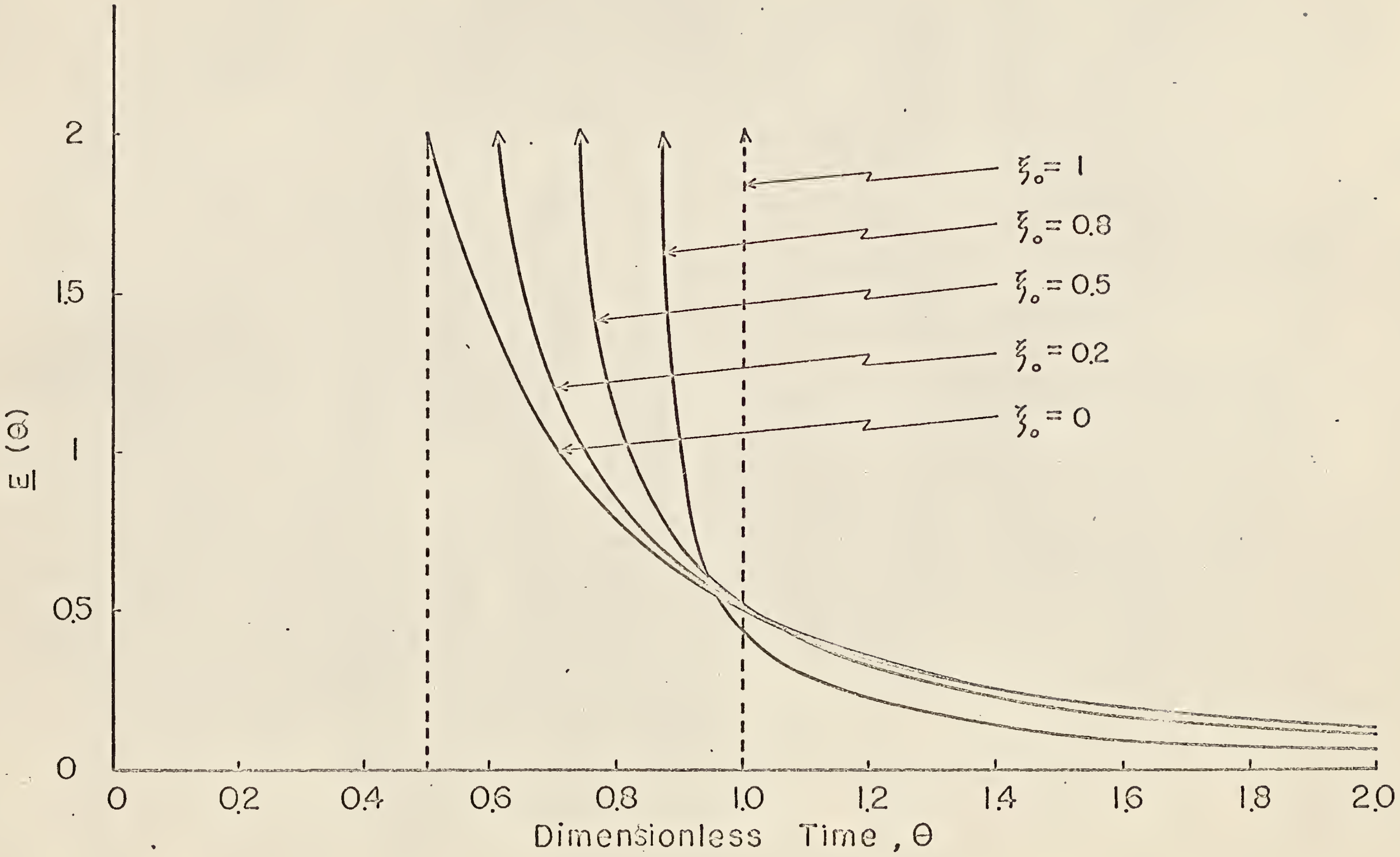

Fig. 4. $\underline{E}$ curves for laminar flow of Bingham model fluids in a cylindrical tube with entrance effect considered and $\gamma' = 0$.

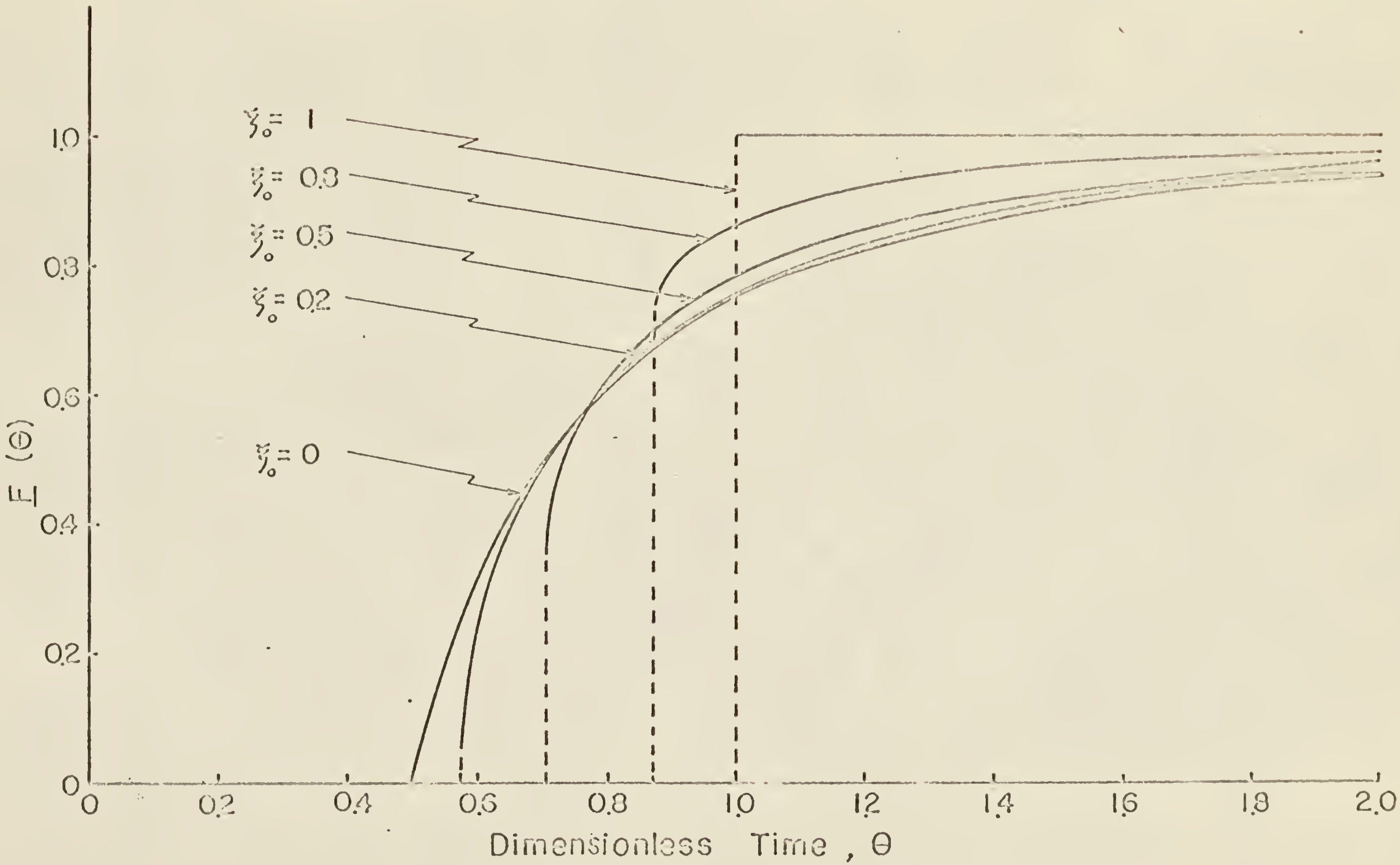

Fig. 5. $\underline{F}$ Curves for laminar flow of Bingham model fluids in a cylidrical tube without entrance effect considered and $\gamma' = 0$.

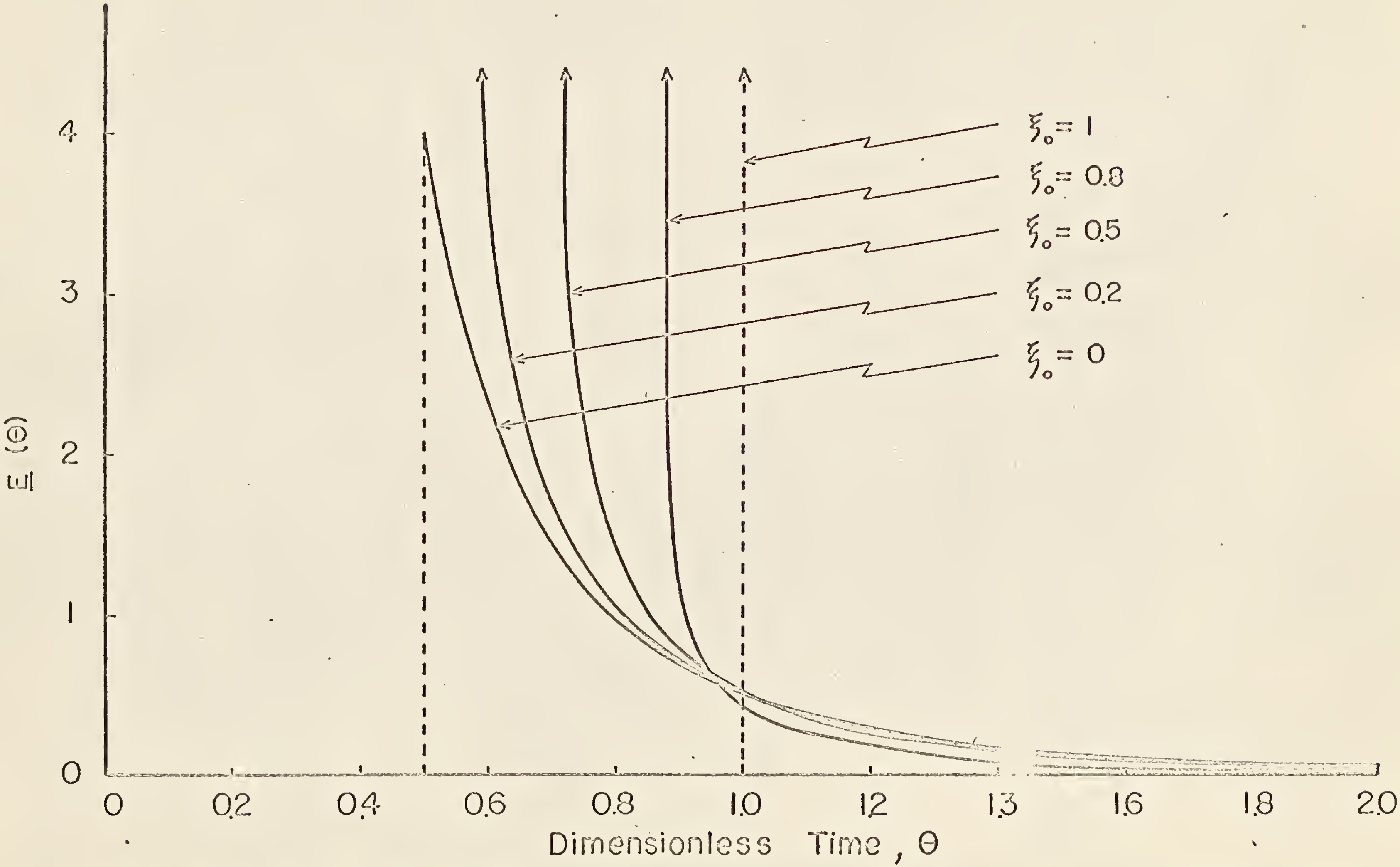

Fig. 6. $\underline{E}$ Curves for laminar flow of Bingham model fluids in a cylindrical tube without entrance effect considere and $\gamma=0$.

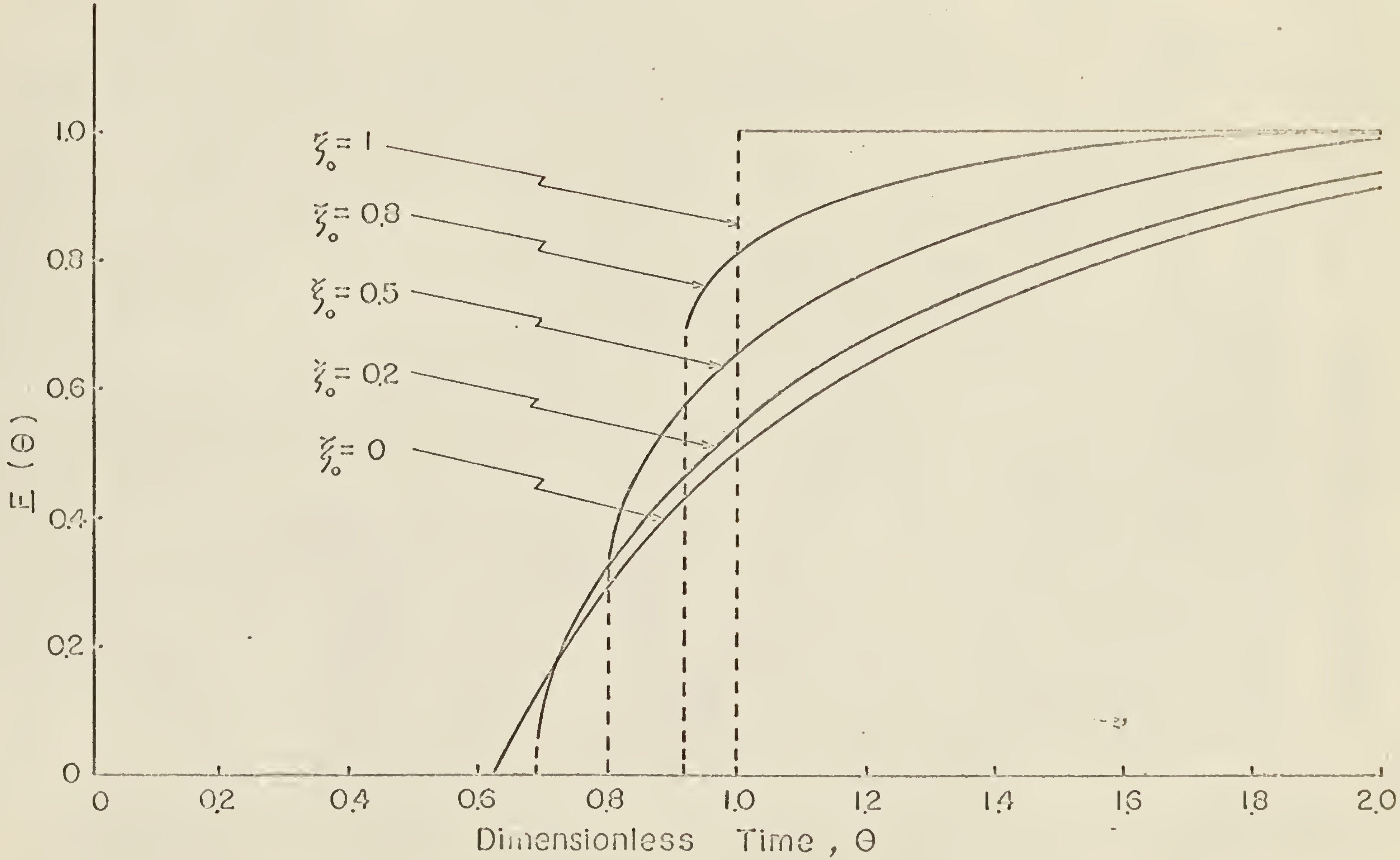

Fig. 7. $\underline{E}$ Curves for laminar flow of Bingham model fluids in a cylindrical tube with entrance effect considered and $\gamma'=0.4$.

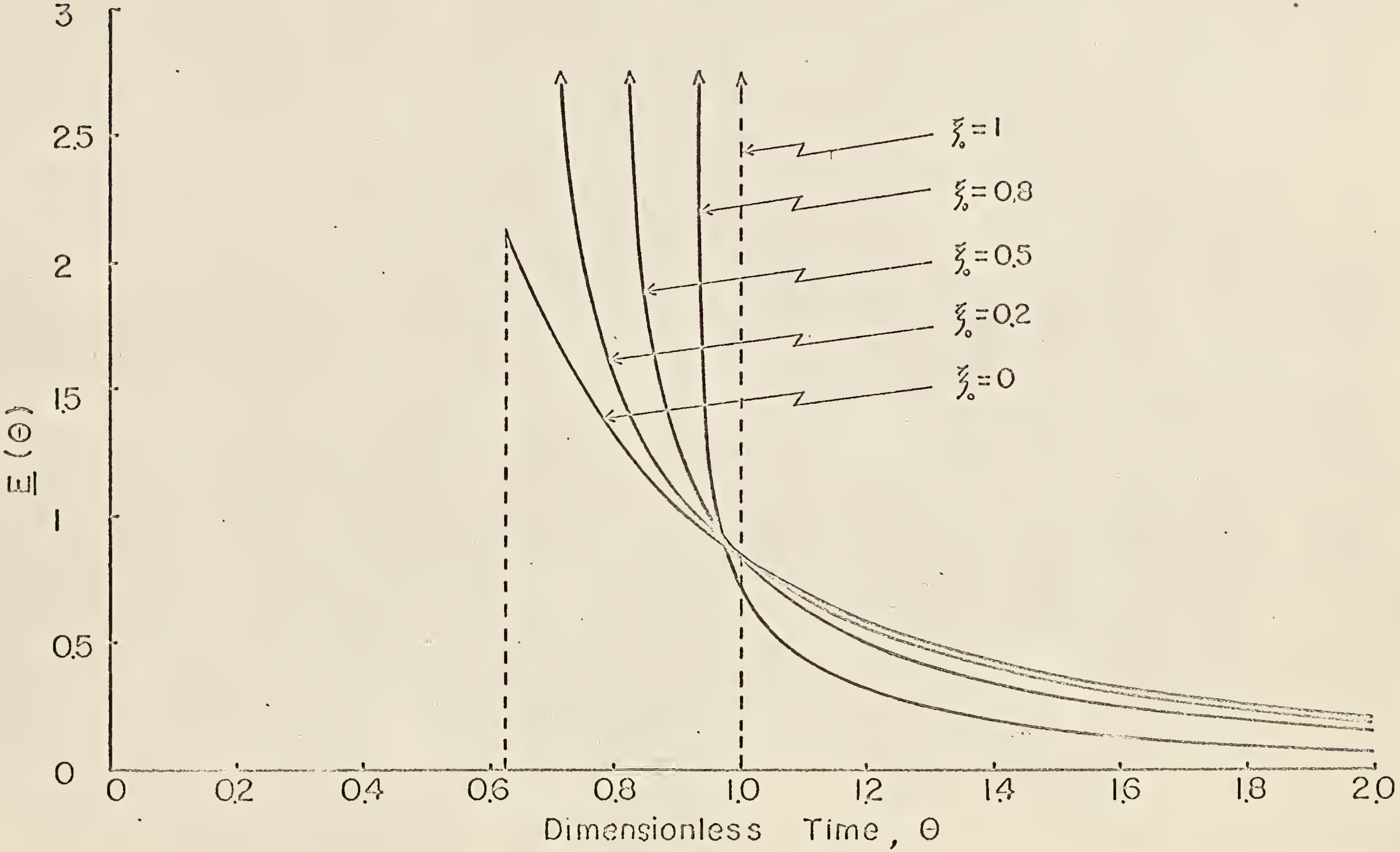

Fig. 8. $\underline{E}$ curves for laminar flow of Bingham model fluids in a cylindrical tube with entrance effect considered and $\alpha = 0.4$.

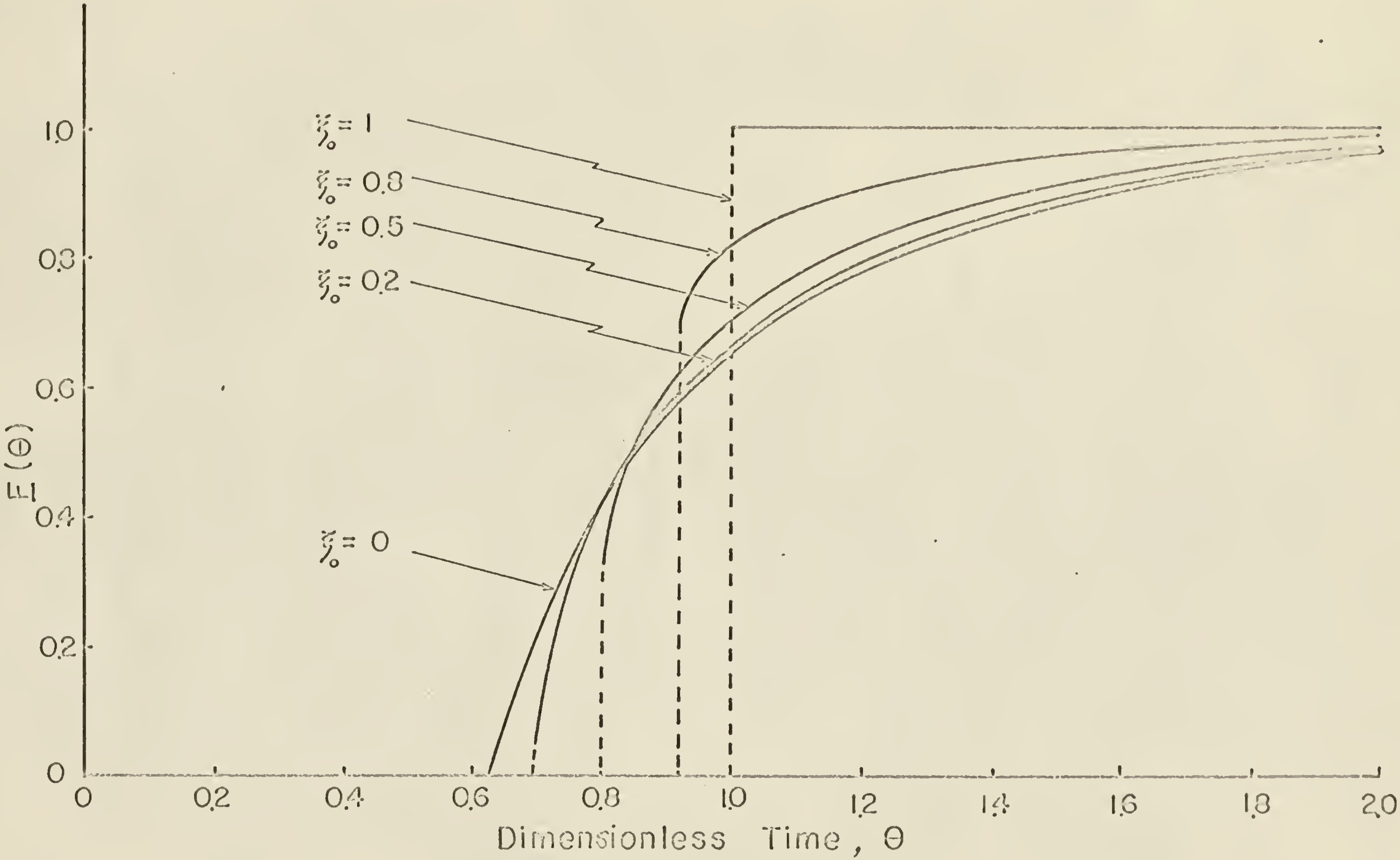

Fig. 9. $\underline{F}$ Curves for laminar flow of Bingham model fluid in a cylindrical tube without entrance effect considered and $\gamma = 0.4$.

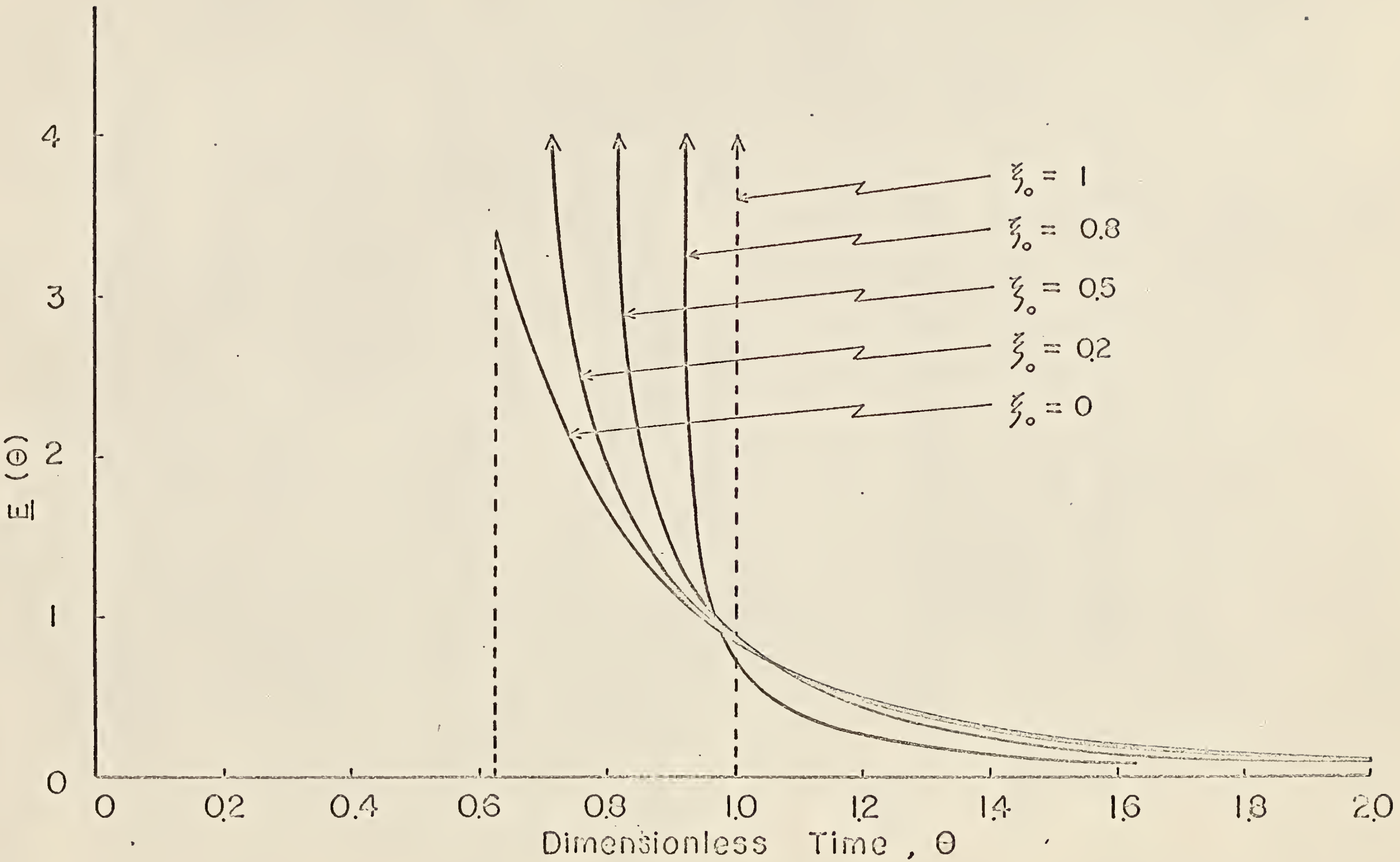

Fig. 10. $\underline{E}$ curves for laminar flow of Bingham model fluids in a cylindrical tube without entrance effect considered and $\gamma=0.4$.

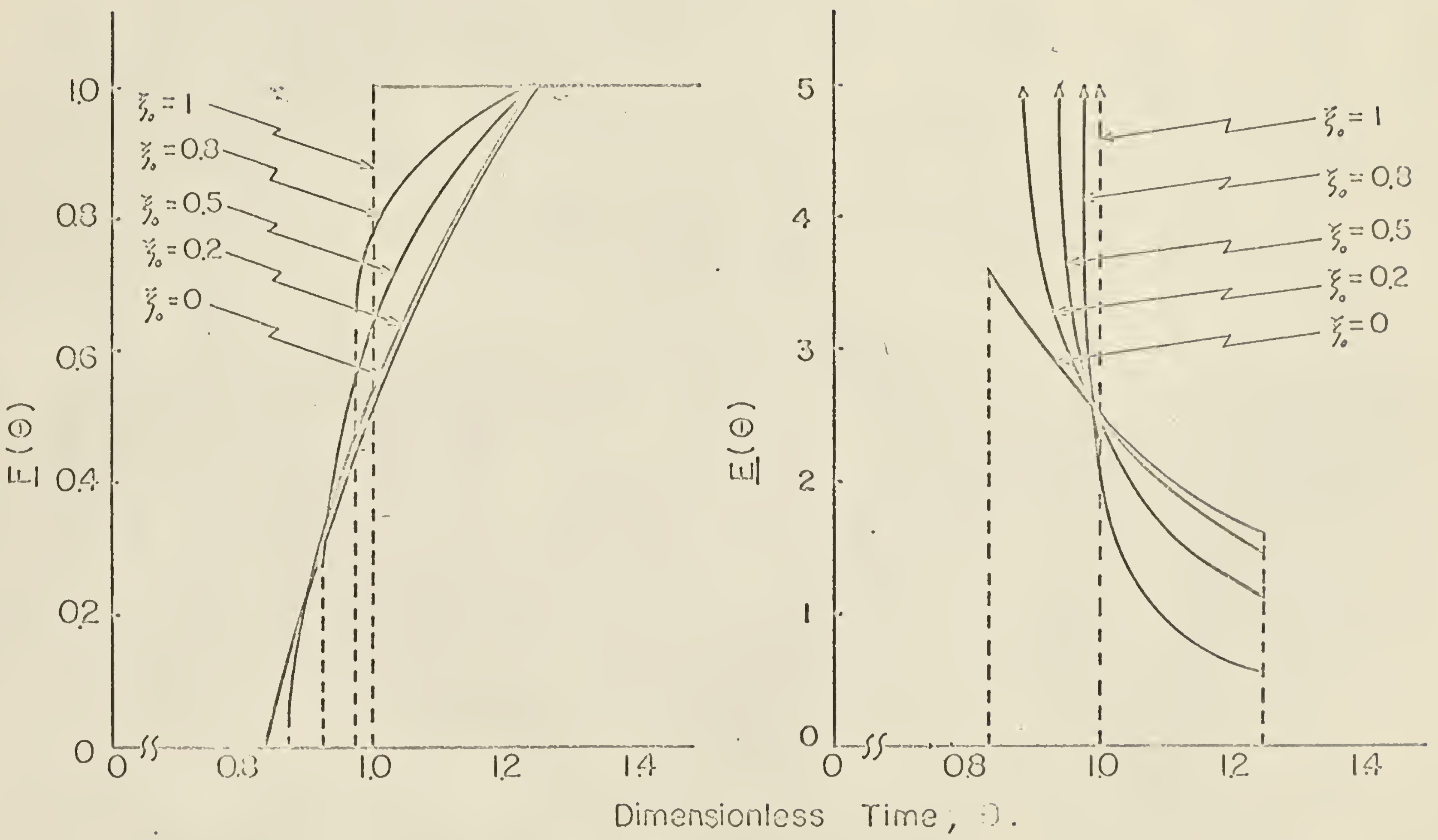

Fig. 11. F and E curves for laminar fluid of Bingham model fluids in a cylindrical tube with entrance effect considered and $\gamma = 0.8$.

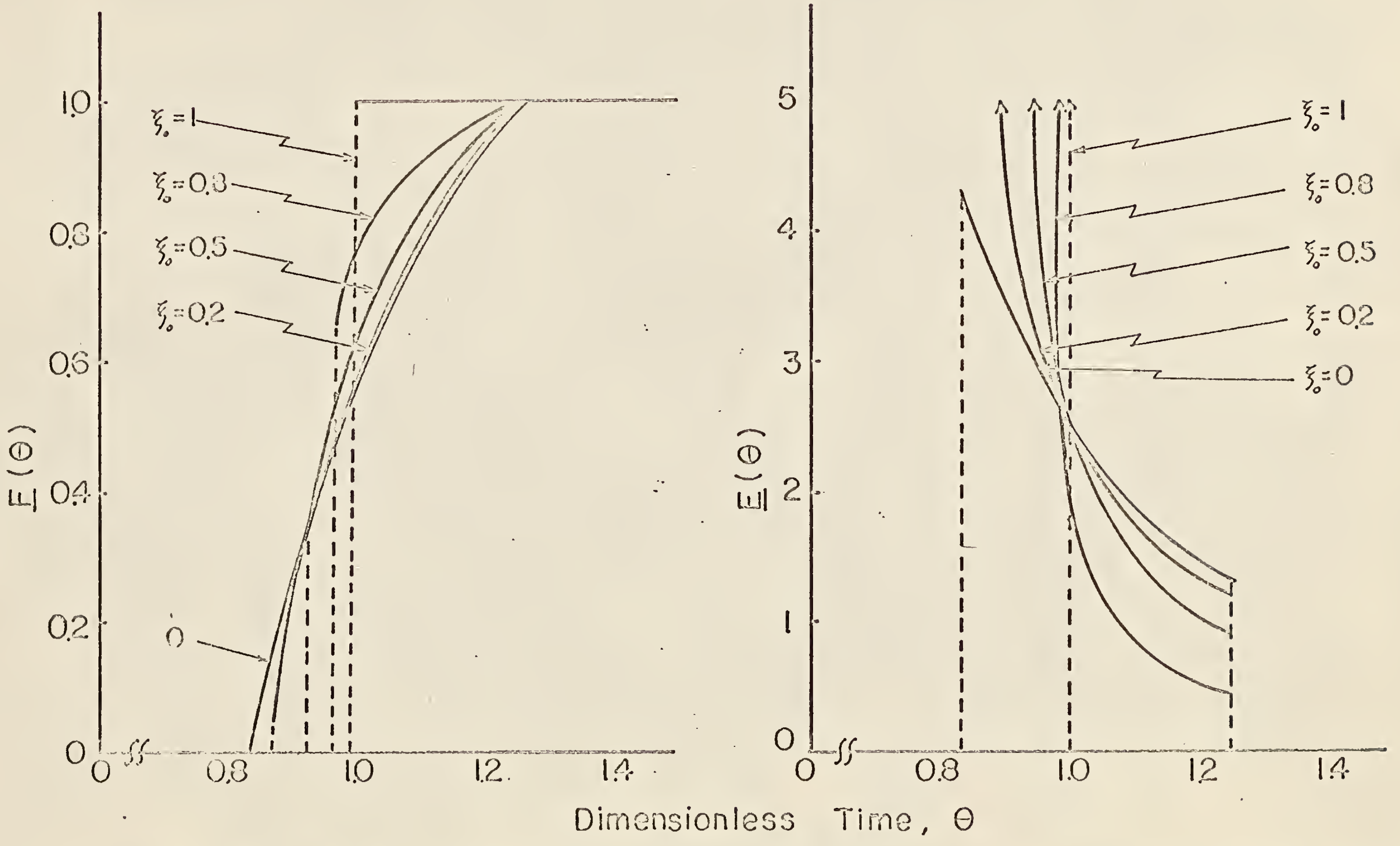

Fig. 12. F and E curves for laminar flow of Bingham model fluid in a cylindrical tube without entrance considered and $\gamma=0.8$.

rheological behavior of a Bingham plastic fluid and the slip velocity at the tube wall. When ξ_0 becomes 0, the flow system reduces to the Newtonian case and the resulting concentration distribution are analogous with those proposed by other investigators (4,5). However, the E-curve and F-curve are deformed in actual observations since the effect of molecular diffusion does exist in actual systems.

IV. DISPERSION MODEL

A generalized mathematical expression for diffusion in the flow system at constant temperature and pressure has been given (2) where D, the molecular diffusion coefficient, is assumed to be constant. Suppose that there is no chemical reaction occurring in a fluid which flows through a circular tube and that the concentration distribution is symmetrical about the central line of the tube. The diffusion equation becomes (see Fig. (2))

$$\frac{\partial C}{\partial t} = D\left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial x^2}\right) - V_x(r)\frac{\partial C}{\partial x} \tag{4-1}$$

Taylor (8) has shown that under the condition

$$E >> D \tag{4-2}$$

the transfer of C along the tube by molecular diffusion is small compared with that produced by convection and thus the $\frac{\partial^2 C}{\partial x^2}$ term may be neglected. He has also shown that, when the time necessary for a radial variation in C to die down owing to radial diffusion is much shorter than the time necessary for an appreciable change in C to occur through longitudinal convection, an approximate solution can be obtained after certain simplifications. The model corresponding to such a solution of the partial differential equation of diffusion is called the dispersion model.

Taylor's assumption can also be expressed as (6)

$$\frac{L}{V_m} >> \frac{R^2}{(3.8)^2 D} \tag{4-3}$$

This is applicable to all types of flow models as long as fluids flow through circular tubes and the $\frac{\partial C}{\partial x}$ term can be taken as zero in the derivation (6).

If rectangular coordinates are used (see Fig. (13)), a corresponding simplified expression of the partial differential equation for diffusion is

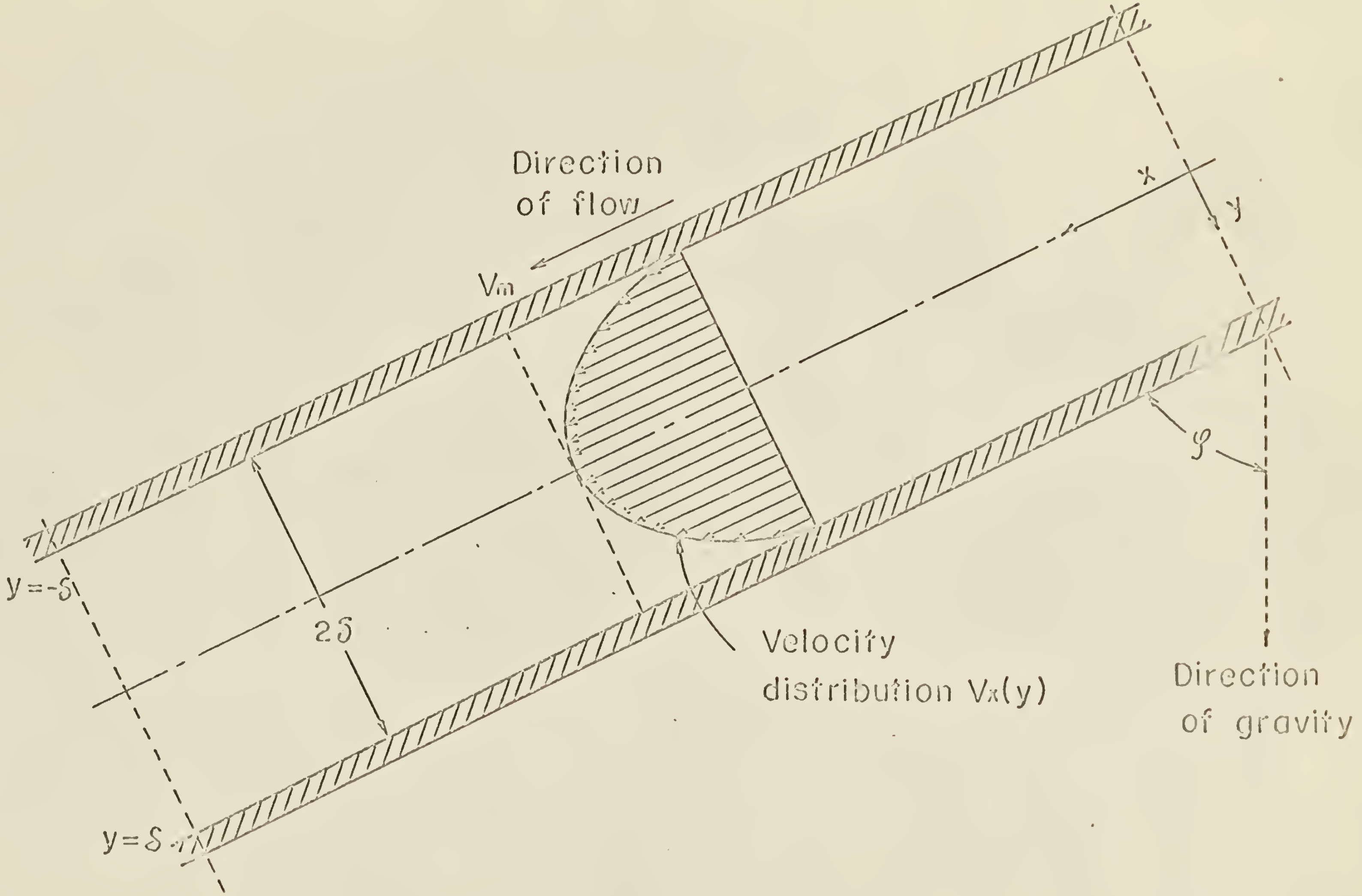

Fig. 13. Flow of a viscous fluid in a slit (openning between two parallel plates).

$$\frac{\partial C}{\partial t} = D\,\frac{\partial^2 C}{\partial y^2} - V_x(y)\,\frac{\partial C}{\partial x} \qquad (4\text{-}4)$$

In order to find the conditions under which Taylor's limiting condition may be valid, it is necessary to calculate how rapidly concentration becomes uniform due to molecular diffusion. The governing differential equation is

$$\frac{\partial C}{\partial t} = D\,\frac{\partial^2 C}{\partial y^2} \qquad (4\text{-}5)$$

subject to the boundary conditions

$$\frac{\partial C}{\partial y} = 0 \qquad \text{at} \quad y = 0 \text{ and } \pm\delta \qquad (4\text{-}6)$$

It should be noted that the velocity distribution of the fluid over the cross section does exist. Introducing the dimensionless variables

$$\theta = \frac{t\bar{V}_x}{L} = \frac{t}{\bar{t}}\,; \qquad \zeta = \frac{y}{\delta}$$

one obtains Equation (5) in dimensionless form as

$$\frac{\partial C}{\partial \theta} = \frac{D\bar{t}}{\delta^2}\cdot\frac{\partial^2 C}{\partial \zeta^2} \qquad (4\text{-}7)$$

The technique of separation of variables is used to solve this equation, or in other words, $C = F(\theta)G(\zeta)$. Substitution of this relation into Equation (7) gives

$$\frac{1}{F}\,\frac{dF}{d\theta} = \frac{D\bar{t}}{\delta^2}\,\frac{1}{G}\cdot\frac{d^2G}{d\theta^2} = -\alpha^2 \qquad (4\text{-}8)$$

This equation has the following solutions:

$$F(\theta) = A_1\, e^{-\alpha^2\theta} \qquad (4\text{-}9)$$

$$G(\zeta) = A_2 \cos\left(\sqrt{\frac{\alpha^2\delta^2}{D\bar{t}}}\,\zeta\right) + A_3 \sin\left(\sqrt{\frac{\alpha^2\delta^2}{D\bar{t}}}\,\zeta\right) \qquad (4\text{-}10)$$

in which A_1, A_2 and A_3 are constants to be determined by the boundary conditions. A_3 must be zero because the concentration distribution is symmetrical around the center line of the tube. Equation (6) requires that

$$\sin \sqrt{\frac{\alpha^2 \delta^2}{D\bar{t}}} = 0$$

Therefore,

$$\sqrt{\frac{\alpha^2 \delta^2}{D\bar{t}}} = n\pi , \qquad n = 0,\ 1,\ \ldots,\ n \tag{4-11}$$

The root of this equation corresponding to the lowest value of α is

$$\alpha_1^2 = \pi^2 \frac{D\bar{t}}{\delta^2} \tag{4-12}$$

and the solution of Equation (7) takes the form

$$C = A\, e^{-\alpha_1^2 \theta} \cos \left(\sqrt{\frac{\alpha^2 \delta^2}{D\bar{t}}}\, \zeta \right) \tag{4-13}$$

The fact that the concentration at $\theta = 0$ and $\zeta = 0$ is 1 ensures that $A = 1$. From Equation (13), the time necessary to decay the central concentration down to e^{-1} of its initial value is

$$t_1 = \frac{\delta^2}{D\pi^2} = 0.101 \frac{\delta^2}{D} \tag{4-14}$$

which implies that

$$\frac{L}{V_m} >> 0.101 \frac{\delta^2}{D} \tag{4-15}$$

in order that Taylor's second limiting condition may be applicable. Equation (15) means that the time necessary for appreciable effects to appear, owing to convective transport, is long compared with cross-sectional variations of concentration through the action of the molecular diffusion.

The systems studied in this chapter are those in which the dispersion in steady flow is due to the combined action of convection parallel to the plates and molecular diffusion in the direction perpendicular to the plates. Restrictions suggested by Taylor are basically followed in the present analysis.

FLOW THROUGH A SLIT

1. Bingham plastic model

The diffusion equation becomes

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial y^2} - V(y) \frac{\partial C}{\partial x} \qquad (4\text{-}4)$$

When a one-dimensional rheological statement is valid, the velocity distribution of the Bingham plastic fluid in rectangular coordinates is

$$V_x(y) = V_m \left[1 - \frac{(\frac{y}{\delta} - \zeta_0)^2}{(1 - \zeta_0)^2} \right], \qquad \text{for } |y| \geq |y_0| \quad (3\text{-}14)$$

where

$$V_m = \frac{\rho g \delta^2 \cos\varphi}{2\mu_0} (1 - \zeta_0)^2 \qquad (3\text{-}13)$$

and

$$\overline{V}_x = \frac{2}{3} \beta V_m \qquad (3\text{-}17)$$

Combining Equations (4) and (3-14) yields

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial y^2} - V_m \left[1 - \frac{((\frac{y}{\delta}) - \zeta_0)^2}{(1 - \zeta_0)^2} \right] \frac{\partial C}{\partial x} \qquad \text{for } |y| \geq |y_0|$$

$$(4\text{-}18)$$

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial y^2} - V_m \frac{\partial C}{\partial x} \qquad \text{for } |y| < |y_0|$$

Introducing the dimensionless variables

$$\theta = \frac{t\bar{V}_x}{L} = \frac{t}{\bar{t}} \quad ; \qquad \zeta = \left|\frac{y}{\delta}\right| ; \quad \eta = \frac{x}{L}$$

one obtains Equation (18) in dimensionless form as

$$\frac{\partial c}{\partial \theta} = \frac{D\bar{t}}{\delta^2} \cdot \frac{\partial^2 c}{\partial \zeta^2} - \frac{3}{2\beta}\left[1 - \frac{(\zeta - \zeta_0)^2}{(1 - \zeta_0)^2}\right] \frac{\partial c}{\partial \eta} \qquad \text{for } \zeta \geq \zeta_0$$

$$= \frac{D\bar{t}}{\delta^2} \cdot \frac{\partial^2 c}{\partial \zeta^2} - \frac{3}{2\beta} \cdot \frac{\partial c}{\partial \eta} \qquad \text{for } \zeta < \zeta_0 \qquad (4\text{-}19)$$

with the boundary conditions

$$\frac{\partial c}{\partial \zeta} = 0 \qquad \text{at} \quad \zeta = 0 \text{ and } 1 \qquad (4\text{-}20)$$

since the plates are impermeable. For convenience the derivative following the mean speed of flow is introduced. Relative to these axes, the velocity distribution is

$$V_{x1} = V_x - \bar{V}_x$$

$$= \left[\left(\frac{3}{2\beta} - 1\right) - \frac{3}{2\beta} \; \frac{(\zeta - \zeta_0)^2}{(1 - \zeta_0)^2} \right] \bar{V}_x \qquad \text{for } \zeta \geq \zeta_0$$

$$= \left(\frac{3}{2\beta} - 1\right) \bar{V}_x \qquad \text{for } \zeta < \zeta_0 \qquad (4\text{-}21)$$

or

$$V_{x1} = \frac{\bar{V}_x}{(\zeta_0 + 2)(1 - \zeta_0)^2} \left[(1 - \zeta_0)^3 - 3(\zeta - \zeta_0)^2 \right] \qquad \text{for } \zeta \geq \zeta_0$$

$$\frac{\bar{V}_x}{(\zeta_0 + 2)(1 - \zeta_0)^2} \; (1 - \zeta_0)^3 \qquad \text{for } \zeta < \zeta_0 \qquad (4\text{-}22)$$

Let $\eta_1 = \eta - \theta$, Equation (19) becomes

$$\frac{\partial C}{\partial \theta} = \frac{D\bar{t}}{\delta^2}\frac{\partial^2 C}{\partial y^2} - \frac{1}{(\zeta_0+2)(1-\zeta_0)^2}\left[(1-\zeta_0)^3 - 3(\zeta-\zeta_0)^2\right]\frac{\partial C}{\partial \eta_1} \quad \text{for } \zeta \geq \zeta_0$$

$$= \frac{D\bar{t}}{\delta^2}\frac{\partial^2 C}{\partial y^2} - \frac{1}{(\zeta_0+2)(1-\zeta_0)^2}(1-\zeta_0)^3\frac{\partial C}{\partial \eta_1} \quad \text{for } \zeta < \zeta_0 \tag{4-23}$$

Since the moving axes which move with mean fluid velocity are introduced, the transfer of tracer concentration across the plane at which η_1 is constant depends only on the cross-sectional variation of tracer concentration. A partial equilibrium may be assumed for a small cross-sectional variation in C. If $\frac{\partial C}{\partial \eta_1}$ is considered to be independent of ζ, the small cross-sectional variation in C can be calculated from

$$\frac{D\bar{t}}{\delta^2}\frac{\partial^2 C}{\partial \zeta^2} = \frac{1}{(\zeta_0+2)(1-\zeta_0)^2}\left[(1-\zeta_0)^3 - 3(\zeta-\zeta_0)^2\right]\frac{\partial C}{\partial \eta_1} \quad \text{for } \zeta \geq \zeta_0$$

$$= \frac{1}{(\zeta_0+2)(1-\zeta_0)^2}(1-\zeta_0)^3\frac{\partial C}{\partial \eta_1} \quad \text{for } \zeta < \zeta_0 \tag{4-24}$$

The concentration distribution is found as

$$C = C_0 + \frac{1}{2(\zeta_0+2)(1-\zeta_0)^2}\frac{\delta^2}{D\bar{t}}\left(\frac{\partial C}{\partial \eta_1}\right)\left[(1-\zeta_0)^3\zeta^2 - \frac{1}{2}(\zeta-\zeta_0)^4\right. \quad \text{for } \zeta \geq \zeta_0$$

$$= C_0 + \frac{1}{2(\zeta_0+2)(1-\zeta_0)^2}\frac{\delta^2}{D\bar{t}}\left(\frac{\partial C}{\partial \eta_1}\right)(1-\zeta_0)^3\zeta^2 \quad \text{for } \zeta < \zeta_0 \tag{4-25}$$

The rate to which C is transported across a section at η_1 is

$$Q = 2\delta W \int_0^1 CV_x(\zeta)\, d\zeta \tag{4-26}$$

Therefore

$$\frac{Q}{2\delta W} = \int_0^{\zeta_0} \left[C_0 + M(1-\zeta_0)^3 \zeta^2 \right] N(1-\zeta_0)^3 \, d\zeta$$

$$+ \int_0^1 \left[C_0 + M\left\{ (1-\zeta_0)^3\zeta^2 - \tfrac{1}{2}(\zeta-\zeta_0)^4 \right\} \right] N\left[(1-\zeta_0)^3 - 3(\zeta-\zeta_0)^2 \right] d\zeta \qquad (4\text{-}27)$$

where

$$M = \frac{\delta^2}{D\bar{t}} \cdot \frac{\partial C}{\partial \eta_1} \cdot \frac{1}{2(\zeta_0 + 2)(1-\zeta_0)^2}$$

$$N = \frac{\bar{V}_x}{(\zeta_0 + 2)(1-\zeta_0)^2}$$

and

$$\frac{Q}{2\delta W} = -\frac{8}{105}\left(1 + \frac{33}{16}\zeta_0 + \frac{21}{16}\zeta_0^2\right)\left(\frac{1-\zeta_0}{\zeta_0+2}\right)^2 \frac{\delta^2 \bar{V}_x^2}{DL} \frac{\partial C_{\eta 1}}{\partial \eta_1} \qquad (4\text{-}28)$$

Since the cross-sectional variations in C are assumed to be small compared with those in the longitudinal direction, $\partial C_{\eta 1}/\partial \eta_1$ is nearly the same as $\partial C_m/\partial \eta_1$ if C_m is the dimensionless mean concentration over a cross section. Equation (28) may be written as

$$Q = -2\delta W \frac{8}{105}\left(1 + \frac{33}{16}\zeta_0 + \frac{21}{16}\zeta_0^2\right)\left(\frac{1-\zeta_0}{\zeta_0+2}\right)^2 \frac{\delta^2 \bar{V}_x^2}{DL} \frac{\partial C_m}{\partial \eta_1} \qquad (4\text{-}29)$$

Comparing this with Fick's law of diffusion, it can be seen that C_m is dispersed relative to a plane which moves with velocity $\bar{V}_x$ with a diffusion coefficient

$$E = \frac{8}{105}\left(1 + \frac{33}{16}\zeta_0 + \frac{21}{16}\zeta_0^2\right)\left(\frac{1-\zeta_0}{\zeta_0+2}\right)^2 \frac{\delta^2 \bar{V}_x^2}{D} \qquad (4\text{-}30)$$

which is called the effective or apparent dispersion coefficient.

It should be remembered that two conditions, Equations (2) and (15), are necessary for the given approximate solution of the equation for diffusion to be valid to interpret the longitudinal dispersion of a solute in a stream. From the first condition and Equation (30), one knows that

$$\frac{8}{105}\left(1 + \frac{33}{16}\zeta_0 + \frac{21}{16}\zeta_0^2\right)\left(\frac{1-\zeta_0}{\zeta_0+2}\right)^2 \frac{\delta^2 \bar{V}_x^2}{D} >> D \qquad (4\text{-}31)$$

in order that the longitudinal molecular diffusion may be negligible compared with the dispersive effect represented by E. The second condition ensures that the time necessary for a cross-sectional variation in C to die down owing to cross-sectional diffusion is much shorter than the time necessary for any appreciable change in C to occur through longitudinal convection. From this condition and the fact that

$$V_m = \frac{3}{\zeta_0 + 2} \bar{V}_x \qquad (4\text{-}32)$$

one obtains

$$\frac{L}{\bar{V}_x} >> 0.034\ (\zeta_0 + 2)\ \frac{\delta^2}{D} \qquad (4\text{-}33)$$

Combining Equations (31) and (33) gives the conditions under which the given solution is valid.

2. Ellis Model

It has been shown that the momentum flux is related to the velocity gradient according to (2)

$$\tau_{yx} = \left(\frac{P_0 - P_L}{L}\right) y \qquad (4\text{-}34)$$

and the fluid which has the shear-stress and shear-rate relation

$$-\frac{dV_x}{dy} = (\varphi_0 + \varphi_1 |\tau_{yx}|^{n-1})\ \tau_{yx} \qquad (4\text{-}35)$$

is called the Ellis model. φ_0, φ_1 and n are measurable parameters. The Ellis model is the simplest and most generally useful three-constant model. It describes properly the lower limiting viscosity η_0. This model seems to have sufficient flexibility to fit the data for various types of fluids. Substituting Equation (34) into Equation (35), one obtains

$$-\frac{dV_x}{dy} = \left\{\varphi_0 + \varphi_1 \left[(\frac{-\Delta P}{L})y \right]^{n-1} \right\} (\frac{-\Delta P}{L})y$$

$$= \varphi_0 (\frac{-\Delta P}{L})y + \varphi_1 (\frac{-\Delta P}{L})^n \cdot y^n \tag{4-36}$$

This equation can be integrated subject to the boundary condition

$V_x = 0$, at $y = \pm \delta$

The resulting velocity distribution is

$$V_x = \varphi_0 (\frac{-\Delta P}{L}) \frac{\delta^2}{2} \left[1 - (\frac{y}{\delta})^2\right] + \varphi_1 (\frac{-\Delta P}{L})^n \frac{\delta^{n+1}}{n+1} \left[1 - (\frac{y}{\delta})^{n+1}\right] \tag{4-37}$$

The maximum velocity is the velocity at y = 0, that is,

$$V_m = \varphi_0 (\frac{-\Delta P}{L}) \frac{\delta^2}{2} + \varphi_1 (\frac{-\Delta P}{L})^n \frac{\delta^{n+1}}{n+1}$$

$$= V_{m1} + V_{m2} \tag{4-38}$$

Expressing $\bar{V}_x$ and V_x in terms of V_{m1} and V_{m2}, one obtains

$$\bar{V}_x = \frac{2}{3} V_{m1} + \frac{n+1}{n+2} V_{m2} \tag{4-39}$$

and

$$V_x = V_{m1}\left[1 - (\frac{y}{\delta})^2\right] + V_{m2}\left[1 - (\frac{y}{\delta})^{n+1}\right] \tag{4-40}$$

Combining Equations (40) and (4) yields the governing partial differential equation, that is

$$\frac{\partial C}{\partial \theta} = \frac{D\bar{t}}{\delta^2}\frac{\partial^2 C}{\partial \zeta^2} - \frac{1}{\bar{V}_x}\left[V_{m1}\,(1-\zeta^2) + V_{m2}\,(1-\zeta^{n+1})\right]\left(\frac{\partial C}{\partial \eta}\right) \qquad (4\text{-}41)$$

with the boundary conditions

$$\frac{\partial C}{\partial \zeta} = 0\,, \qquad \text{at} \quad \zeta = 0 \text{ and } 1 \qquad (4\text{-}42)$$

If one defines concentration and velocity relative to axes which move with the mean velocity of fluid flow, the velocity distribution is

$$\begin{aligned} V_{x1} &= V_x - \bar{V}_x \\ &= V_{m1}\,(\tfrac{1}{3} - \zeta^2) + V_{m2}\,(\tfrac{1}{n+2} - \zeta^{n+1}) \end{aligned} \qquad (4\text{-}43)$$

and the transport equation becomes

$$\frac{\partial C}{\partial \theta} = \frac{D\bar{t}}{\delta^2}\left(\frac{\partial^2 C}{\partial \zeta^2}\right) - \frac{1}{\bar{V}_x}\left[V_{m1}\,(\tfrac{1}{3} - \zeta^2) + V_{m2}\,(\tfrac{1}{n+2} - \zeta^{n+1})\right]\left(\frac{\partial C}{\partial \eta_1}\right) \qquad (4\text{-}44)$$

The distribution of C under the conditions, $\partial C/\partial \theta = 0$ and $\partial C/\partial \eta_1 =$ constant, is found to be in the form

$$C = C_0 + \frac{\delta^2}{D\bar{t}\bar{V}_x}\frac{\partial C}{\partial \eta_1}\left\{\frac{1}{6}V_{m1}\zeta^3 - \frac{1}{12}V_{m1}\zeta^4 + V_{m2}\left[\frac{\zeta^2}{2(n+2)} - \frac{\zeta^{n+3}}{(n+2)(n+3)}\right]\right\} \qquad (4\text{-}45)$$

where C_0 is the concentration of the tube at $\zeta = 0$.

The rate at which C is transported across a section is

$$Q = 2W\delta\int_0^1 CV_{x1}d\zeta \qquad (4\text{-}46)$$

Inserting values of C and V_{x1} into Equation (46), Q is found to be

$$Q = -2\delta W \frac{\delta^2}{D\bar{t}\bar{V}_x} \left(\frac{\partial C}{\partial \eta_1}\right) \left[\frac{8}{945} V_{m1}^2 + \frac{8}{27(n+4)(2n+5)} V_{m2}^2 + \frac{16(n+1)(n+9)}{405(n+2)(n+4)(n+6)} V_{m1} V_{m2} \right] \qquad (4\text{-}47)$$

Since $\partial C/\partial \eta_1$ has been assumed to be independent of ζ, C may be replaced by C_m. The rate of transfer of matter in a slit due to a diffusivity E is

$$Q = -\frac{2\delta W E}{L} \frac{\partial C_m}{\partial \eta_1} \qquad (4\text{-}48)$$

Comparing Equations (47) and (48), it can be seen that

$$E = \frac{\delta^2}{D} \left[\frac{8}{945} V_{m1}^2 + \frac{8}{27(n+4)(2n+5)} V_{m2}^2 + \frac{16(n+1)(n+9)}{405(n+2)(n+4)(n+6)} V_{m1} V_{m2} \right] \qquad (4\text{-}49)$$

It should be recalled that

$$E >> D$$

in order that the axial molecular diffusion be negligible and that the following condition must be satisfied so that Taylor's second assumption may be true:

$$\frac{L}{V_m} >> 0.101 \frac{\delta^2}{D}$$

where

$$V_m = \varphi_0 \left(\frac{-\Delta P}{L}\right) \frac{\delta^2}{2} + \varphi_1 \left(\frac{-\Delta P}{L}\right)^n \frac{\delta^{n+1}}{n+1} \qquad (4\text{-}50)$$

When φ_1 is zero the Ellis model reduces to the Newtonian case and the effective dispersion coefficient becomes

$$E = \frac{2}{105} \frac{\delta^2 \bar{V}_x^2}{D} \tag{4-51}$$

because V_{m2} becomes zero. For the case $\varphi_0 = 0$, V_{m1} becomes zero, that is, for an Ostwald-de Waele model, the effective dispersion coefficient is

$$E = \frac{2}{3(n+4)(2n+5)} \frac{\delta^2 \bar{V}_x^2}{D} \tag{4-52}$$

FLOW THROUGH CYLINDRICAL TUBES

If the transfer of a tracer along the tube by the molecular diffusion is small compared with that produced by convection, the transport equation becomes (see Fig. (2))

$$\frac{\partial C}{\partial t} = D \left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r} \frac{\partial C}{\partial r}\right) - V_x(r) \frac{\partial C}{\partial x} \tag{4-1}$$

The boundary conditions are

$$\frac{\partial C}{\partial r} = 0 \qquad \text{at} \quad r = 0 \text{ and } R \tag{4-53}$$

because the concentration distribution is symmetrical around the tube and the wall of the tube is impermeable. Introducing the following dimensionless variables

$$\theta = \frac{t}{\bar{t}} = \frac{t\bar{V}_x}{L}, \qquad \xi = \frac{r}{R}, \qquad \eta = \frac{x}{L}$$

Equations (1) and (53) become

$$\frac{\partial C}{\partial \theta} = \frac{D\bar{t}}{R^2} \left(\frac{\partial^2 C}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial C}{\partial \xi}\right) - \frac{V_x}{\bar{V}_x} \frac{\partial C}{\partial \eta} \tag{4-54}$$

and

$$\frac{\partial C}{\partial \xi} = 0 , \qquad \text{at} \quad \xi = 0 \text{ and } 1 \tag{4-55}$$

The condition under which Taylor's approach may be valid can be expressed as (6)

$$\frac{L}{\bar{V}_m} >> \frac{R^2}{(3.8)^2 D} \tag{4-56}$$

Velocity distributions which can be represented by the Bingham plastic and Ellis model will be considered in this section.

1. Bingham Plastic Model

The velocity distribution of Bingham plastic fluid has been derived as (see Equation (3-48))

$$\begin{aligned} V_x &= V_m \left[1 - \frac{(\xi - \xi_0)^2}{(1-\xi_0)^2} \right] & \text{for} \quad \xi \geq \xi_0 \\ &= V_m & \text{for} \quad \xi < \xi_0 \end{aligned} \tag{4-57}$$

and

$$\bar{V}_x = \frac{\alpha}{2} V_m = \frac{1}{6} (3 + 2\xi_0 + \xi_0^2) V_m \tag{4-58}$$

We are now considering the convection across a plane which moves with the mean speed of flow. Writing

$$\eta_1 = \eta - \theta \tag{4-59}$$

the velocity distribution relative to these axes is

$$V_{x1} = V_x - \bar{V}_x$$

$$= \left[\left(\frac{2}{\alpha} - 1\right) - \frac{2}{\alpha} \frac{(\xi - \xi_0)^2}{(1 - \xi_0)^2} \right] \bar{V}_x \qquad \text{for } \xi \geq \xi_0$$

$$(4\text{-}60)$$

$$= \left(\frac{2}{\alpha} - 1\right) \bar{V}_x \qquad \text{for } \xi < \xi_0$$

and Equation (54) becomes

$$\frac{\partial C}{\partial \theta} = \frac{D\bar{t}}{R^2} \left(\frac{\partial^2 C}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial C}{\partial \xi}\right) - \left[\left(\frac{2}{\alpha} - 1\right) - \frac{2}{\alpha} \frac{(\xi-\xi_0)^2}{(1-\xi_0)^2} \right] \frac{\partial C}{\partial \eta_1} \qquad \text{for } \xi \geq \xi_0$$

$$(4\text{-}61)$$

$$= \frac{D\bar{t}}{R^2} \left(\frac{\partial^2 C}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial C}{\partial \xi}\right) - \left(\frac{2}{\alpha} - 1\right) \frac{\partial C}{\partial \eta_1} \qquad \text{for } \xi < \xi_0$$

The small radial variation in C can be calculated from Equation (61) by letting $\partial C/\partial \theta = 0$ and considering $\partial C/\partial \eta_1$ to be independent of ξ, that is,

$$\frac{D\bar{t}}{R^2} \left(\frac{\partial^2 C}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial C}{\partial \xi}\right) = \frac{D\bar{t}}{R^2} \left[\frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial C}{\partial \xi}\right) \right]$$

$$= \left[\left(\frac{2}{\alpha} - 1\right) - \frac{2}{\alpha} \frac{(\xi-\xi_0)^2}{(1-\xi_0)^2} \right] \frac{\partial C}{\partial \eta_1} \qquad \text{for } \xi \geq \xi_0$$

$$(4\text{-}62)$$

$$= \left(\frac{2}{\alpha} - 1\right) \frac{\partial C}{\partial \eta_1} \qquad \text{for } \xi < \xi_0$$

Equation (62) has the boundary condition

$$\frac{\partial C}{\partial \xi} = 0 \qquad \text{at } \xi = 0 \text{ and } 1 \qquad (4\text{-}63)$$

The solution of Equation (62) can be obtained as

$$C = C_0 + \frac{R^2}{D\bar{t}} \left(\frac{\partial C}{\partial \eta_1}\right) \int_0^{\xi_0} \frac{1}{\xi} \, d\xi \int_0^{\xi_0} \left(\frac{1}{\alpha} - 1\right) \xi \, d\xi$$

$$+ \frac{R^2}{D\bar{t}} \left(\frac{\partial C}{\partial \eta_1}\right) \int_{\xi_0}^{\xi} \frac{1}{\xi}\, d\xi \int_{\xi_0}^{\xi} \left[\left(\frac{2}{\alpha} - 1\right) - \frac{2}{\alpha} \frac{(\xi-\xi_0)^2}{(1-\xi_0)^2}\right] \xi\, d\xi \qquad \text{for } \xi \geq \xi_0$$

$$= C_0 + \frac{R^2}{D\bar{t}} \left(\frac{\partial C}{\partial \eta_1}\right) \int_{0}^{\xi} \frac{1}{\xi}\, d\xi \int_{\xi_0}^{\xi} \left(\frac{1}{\alpha} - 1\right) \xi\, d\xi \qquad \text{for } \xi < \xi_0$$

$$(4\text{-}64)$$

Inserting $\alpha = \frac{1}{3}(3+2\xi_0+\xi_0^2)$ into Equation (64) yields

$$C = C_0 + \frac{R^2}{D\bar{t}\,\alpha(1-\xi_0)^2} \frac{\partial C}{\partial \eta_1} \left[\left(\frac{1}{4} - \frac{2}{3}\xi_0 - \frac{1}{12}\xi_0^4\right)\xi^2 + \frac{4}{9}\xi_0\xi^3 - \frac{1}{8}\xi^4\right.$$

$$\left. + \frac{13}{72}\xi_0^4 + \frac{1}{6}\xi_0^4 \ln \frac{\xi}{\xi_0}\right] \qquad \text{for } \xi \geq \xi_0$$

$$= C_0 + \frac{R^2}{D\bar{t}\,\alpha(1-\xi_0)^2} \frac{\partial C}{\partial \eta_1} \left[\frac{1}{4} - \frac{2}{3}\xi_0 + \frac{1}{2}\xi_0^2 - \frac{1}{12}\xi_0^4\right]\xi^2, \quad \text{for } \xi < \xi_0$$

$$(4\text{-}65)$$

where C_0 is the value of C at $\xi = 0$.

The rate of transfer of C across the section at η_1 is

$$Q = 2\pi R^2 \left[\int_0^{\xi_0} C V_{m1}\, \xi\, d\xi + \int_{\xi_0}^{1} C V_{x1}(\xi)\, \xi\, d\xi \right. \qquad (4\text{-}66)$$

Substituting Equations (60) and (65) into Equation (66), the rate of transfer of C across the section at η_1 is found to be

$$Q = - \frac{\pi R^4 \bar{V}_x^2}{2LD(3+2\xi_0+\xi_0^2)^2(1-\xi_0)^4} \left[\frac{3}{8} - \frac{44}{35}\xi_0 + \frac{16}{15}\xi_0^2 + \xi_0^4 - \frac{28}{15}\xi_0^5 - \frac{3}{5}\xi_0^6\right.$$

$$+\frac{8}{5}\xi_0^7 - \frac{29}{56}\xi_0^8 + \frac{1}{5}\xi_0^{10} - \xi_0^8 \ln \xi_0\Big]\left(\frac{\partial C}{\partial \eta_1}\right) \qquad (4\text{-}67)$$

The rate of transfer of matter across a cross section with the diffusivity E is

$$Q = -\frac{\pi R^2 E}{L}\left(\frac{\partial C_m}{\partial \eta_1}\right) \qquad (4\text{-}68)$$

It should be noted that $\partial C_m/\partial \eta_1$ is indistinguishable from $\partial C/\partial \eta_1$ as long as $\partial C/\partial \eta_1$ is independent of ξ. Thus it will not be unreasonable to define the effective dispersion coefficient E of Bingham plastic fluids flowing through a cylindrical tube as

$$E = \frac{R^2\bar{V}_x^2}{2D(3+2\xi_0+\xi_0^2)^2(1-\xi_0)^4}\Big[\frac{3}{8} - \frac{44}{35}\xi_0 + \frac{16}{15}\xi_0^2 + \xi_0^4 - \frac{28}{15}\xi_0^5 - \frac{3}{5}\xi_0^6$$

$$+\frac{8}{5}\xi_0^7 - \frac{29}{56}\xi_0^8 + \frac{1}{5}\xi_0^{10} - \xi_0^8 \ln \xi_0\Big] \qquad (4\text{-}69)$$

For the case $\xi_0 = 0$, Equation (69) becomes

$$E = \frac{R^2\bar{V}_x^2}{48D} \qquad (4\text{-}70)$$

This agrees with the result for the Newtonian flow derived by Taylor (6). Since the final expression for E appears to be peculiar, it may be desirable to evaluate the value of E at $\xi_0 = 1$. As $\xi_0 \longrightarrow 1$, $\ln \xi_0$ may be expressed as

$$\ln_{\xi_0 \to 1} \xi_0 = -(1-\xi_0) - \frac{1}{2}(1-\xi_0)^2 - \frac{1}{3}(1-\xi_0)^3 - \frac{1}{4}(1-\xi_0)^4 - \dots \qquad (4\text{-}71)$$

Combining Equations (71) and (69), it can be seen that the effective dispersion coefficient E becomes zero as $\xi_0 \longrightarrow 1$. This is reasonable since we have neglected longitudinal diffusion in the analysis.

The conditions under which the given approximate solution can be used to interpret longitudinal dispersion of a solute are, as stated previously,

$$E >> D \qquad (4\text{-}72)$$

where E is given in Equation (69) and

$$\frac{L}{V_m} = \frac{(3 + 2\xi_0 + \xi_0^2)L}{6\bar{V}_x} >> \frac{R^2}{(3.8)^2 D} \qquad (4\text{-}73)$$

2. Ellis Model

The momentum flux distribution for flow of a fluid through a circular tube is

$$\tau_{rx} = (\frac{P_0 - P_L}{2L})\ r \qquad (4\text{-}74)$$

where $(P_0 - P_L)$ is the result of a pressure gradient and/or gravitational acceleration.

For the case when a one-dimensional rheological statement is valid, the relationship between the shear-stress and shear-rate written in cylindrical coordinates is (see Fig. (2))

$$-\frac{dV_x}{dr} = (\varphi_0 + \varphi_1 |\tau_{rx}|^{n-1})\ \tau_{rx} \qquad (4\text{-}75)$$

where φ_0 ; φ_1 and n are parameters to be determined by experiment.

Substitution of Equation (74) into Equation (75) gives

$$-\frac{dV_x}{dr} = \varphi_0\ (\frac{-\Delta P}{2L})r + \varphi_1 (\frac{-\Delta P}{2L}r)^n \qquad (4\text{-}76)$$

The boundary condition that $V_x = 0$ at $r = R$ can be used to evaluate the integration constant. Then the velocity distribution becomes

$$V_x(r) = \left(\frac{-\Delta P \varphi_0}{2L}\right) \frac{R^2}{2} \left[1-\left(\frac{r}{R}\right)^2\right] + \left(\frac{-\Delta P}{2L}\right)^n \varphi_1 \frac{R^{n+1}}{n+1} \left[1-\left(\frac{r}{R}\right)^{n+1}\right] \qquad (4\text{-}77)$$

The maximum velocity occurs at $r = 0$; thus it has the value

$$V_m = \left(\frac{-\Delta P \varphi_0}{2L}\right) \frac{R^2}{2} + \left(\frac{-\Delta P}{2L}\right)^n \frac{\varphi_1 R^{n+1}}{n+1} \qquad (4\text{-}78)$$

$$= V_{m1} + V_{m2} \qquad (4\text{-}79)$$

Combining Equations (77) and (79) yields

$$V_x = V_{m1}\left[1 - \left(\frac{r}{R}\right)^2\right] + V_{m2}\left[1 - \left(\frac{r}{R}\right)^{n+1}\right] \qquad (4\text{-}80)$$

The average velocity $\bar{V}_x$ is calculated by summing up all the velocity over a cross section and then dividing by the cross-sectional area, which yields

$$\bar{V}_x = \frac{1}{2} V_{m1} + \frac{n+1}{n+3} V_{m2} \qquad (4\text{-}81)$$

The mathematical expression of the diffusion equation for an incompressible fluid flowing in a circular tube with the assumption that the axial diffusion may be neglected is

$$\frac{\partial C}{\partial t} = D\left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r}\right) - V_x \frac{\partial C}{\partial x} \qquad (4\text{-}82)$$

It is convenient in this discussion to define concentration and velocity relative to axes moving with the mean flow. Relative to these axes, the velocity V_{x1} is

$$V_{x1} = V_x - \bar{V}_x = V_{m1}\left[\frac{1}{2} - \left(\frac{r}{R}\right)^2\right] + V_{m2}\left[\frac{2}{n+3} - \left(\frac{r}{R}\right)^{n+1}\right] \qquad (4\text{-}83)$$

Substituting Equation (83) into the governing partial differential equation and writing it in dimensionless variables, one obtains

$$\frac{\partial C}{\partial \theta} = \frac{D\bar{t}}{R}\left(\frac{\partial^2 C}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial C}{\partial \xi}\right) - \frac{1}{\bar{V}_x}\left[V_{m1}\left(\frac{1}{2} - \xi^2\right) + V_{m2}\left(\frac{2}{n+3} - \xi^{n+1}\right)\right]\frac{\partial C}{\partial \eta_1} \qquad (4\text{-}84)$$

The distribution of C in the case when $\partial C/\partial\theta = 0$ and $\frac{\partial C}{\partial \eta_1}$ is independent of ξ is found as

$$C = C_0 + \frac{R^2}{D\bar{t}\,\bar{V}_x}\left(\frac{\partial C}{\partial \eta_1}\right)\left[V_{m1}\left(\frac{1}{8}\xi^2 - \frac{1}{16}\xi^4\right) + \frac{V_{m1}}{n+3}\left(\frac{1}{2}\xi^2 - \frac{1}{n+3}\xi^{n+3}\right)\right] \qquad (4\text{-}85)$$

by applying the boundary conditions that $\partial C/\partial\xi = 0$ at $\xi = 1$.

The rate of transfer at which C is transported across a section at η_1 is

$$Q = 2\pi R^2 \int_0^1 C V_{x1} \xi \, d\xi \qquad (4\text{-}86)$$

Replacing C and V_{x1} by Equations (85) and (83), it is found that

$$Q = 2\pi R^2 \frac{R^2}{D\bar{t}\,\bar{V}_x}\left(\frac{\partial C}{\partial \eta_1}\right)\int_0^1\left[V_{m1}\left(\frac{1}{8}\xi^2 - \frac{1}{16}\xi^4\right) + \frac{V_{m2}}{n+3}\left(\frac{1}{2}\xi^2 - \frac{1}{n+3}\xi^{n+3}\right)\right]$$

$$\left[V_{m1}\left(\frac{1}{2} - \xi^2\right) + V_{m2}\left(\frac{2}{n+3} - \xi^{n+1}\right)\right]\xi \, d\xi$$

$$= -\,2\pi R^2 \frac{R^2}{D\bar{t}\,\bar{V}_x}\left(\frac{\partial C}{\partial \eta_1}\right)\left[\frac{1}{384}V_{m1}^2 + \frac{(n+1)^2}{4(n+3)^3(n+5)}V_{m2}^2\right.$$

$$\left. + \frac{(n+1)(n+11)}{24(n+3)(n+5)(n+7)}V_{m1}V_{m2}\right] \qquad (4\text{-}87)$$

Comparing this with

$$Q = -\eta R^2 \frac{E}{L} \frac{\partial C_{m1}}{\partial \eta_1} \qquad (4\text{-}88)$$

the effective dispersion coefficient E is

$$E = \frac{R^2}{D}\left[\frac{1}{192} V_{m1}^2 + \frac{(n+1)^2}{2(n+3)^3(n+5)} V_{m2}^2 + \frac{(n+1)(n+11)}{12(n+3)(n+5)(n+7)} V_{m1}V_{m2}\right] \qquad (4\text{-}89)$$

It should be noted that the conditions

$$E >> D \qquad (4\text{-}90)$$

and

$$\frac{L}{V_{m1} + V_{m2}} >> \frac{R^2}{(3.8)^2 D} \qquad (4\text{-}91)$$

must be satisfied in order for the analysis to be valid.

When $\varphi_1 = 0$, V_{m2} becomes zero and the Ellis model reduces to the Newtonian case and the effective dispersion coefficient becomes

$$E = \frac{R^2 \bar{V}_x^2}{48D} \qquad (4\text{-}92)$$

If $\varphi_0 = 0$, that is, for the Ostwald-de Waele model, V_{m1} is zero and the effective dispersion coefficient becomes

$$E = \frac{1}{2(n+3)(n+5)} \frac{R^2 \bar{V}_x^2}{D} \qquad (4\text{-}93)$$

which is identical to that derived by Fan and Hwang (19) for the Ostwald-de Waele fluid. The limiting condition for the Ostwald-de Waele model, as derived by Fan and Hwang, are

$$\frac{1}{2(n+3)(n+5)} \frac{R^2 \bar{V}_x^2}{D} >> D \qquad (4\text{-}94)$$

and

$$\frac{L}{\bar{V}_x} \gg \frac{(n+3)R^2}{(n+1)(3.8)^2 D} \qquad (4\text{-}95)$$

The equation of continuity for C_m in the given process is

$$\frac{\partial Q}{\partial \eta_1} = - A\bar{V}_x \frac{\partial C_m}{\partial \theta} \qquad (4\text{-}96)$$

in which A is the cross-sectional area of the system. Substituting for Q from Equation (88) the dispersion relative to axes moving with speed $\bar{V}_x$ is governing by the equation

$$\frac{\partial C_m}{\partial \theta} = \left(\frac{E}{\bar{V}_x L}\right) \frac{\partial^2 C_m}{\partial \eta_1{}^2} \qquad (4\text{-}97)$$

Since the concentration distribution has been reduced to one dimensional statement, one can replace C_m by C in Equation (97), that is,

$$\frac{\partial C}{\partial \theta} = \left(\frac{E}{\bar{V}_x L}\right) \frac{\partial C}{\partial \eta_1{}^2} \qquad (4\text{-}98)$$

This has to be solved subject to the boundary conditions

B.C. 1. $C = 0$ for $\theta = 0$ and $\eta_1 \neq 0$ (4-99)

B.C. 2. $C = \delta(\theta)$ for $\theta = 0$ and $\eta_1 = 0$ (4-100)

B.C. 3. $\int_{-\infty}^{\infty} C \, d\eta_1 = 1$ (4-101)

The solution subject to the given boundary conditions is (20, 21)

$$C = \frac{1}{2\left[\pi\left(\frac{E}{\bar{V}_x L}\right)\theta\right]^{1/2}} \exp\left[- \frac{\eta_1{}^2}{4\left(\frac{E}{\bar{V}_x L}\right)\theta}\right] \qquad (4\text{-}102)$$

Since $\eta_1 = \eta - \theta$ has been defined in this analysis, one may replace η_1 by

$\eta - \theta$, which gives

$$C = \frac{1}{2\left[\pi\left(\frac{E}{\bar{V}_x L}\right)\theta\right]^{1/2}} \exp\left[-\frac{(\eta-\theta)^2}{4\left(\frac{E}{\bar{V}_x L}\right)\theta}\right] \qquad (4\text{-}103)$$

An experiment designed by setting $\eta = 1$ yields the outlet tracer concentration distribution function to be

$$C = \underline{E}(\theta)$$

$$= \frac{1}{2\left[\pi\left(\frac{E}{\bar{V}_x L}\right)\theta\right]^{1/2}} \exp\left[-\frac{(1-\theta)^2}{4\left(\frac{E}{\bar{V}_x L}\right)\theta}\right] \qquad (4\text{-}104)$$

where $E/\bar{V}_x L$ is a function which varies with dimensionless parameter $\delta^2\bar{V}_x^2/D$ (or $R^2\bar{V}_x^2/D$) and the flow behavior index describing different flow models. Integrating the given distribution function from 0 to θ, we can obtain the concentration distribution corresponding to a step function input, that is,

$$C = \underline{F}(\theta)$$

$$= \int_0^{\theta} \frac{1}{2\left[\pi\left(\frac{E}{\bar{V}_x L}\right)\theta\right]^{1/2}} \exp\left[-\frac{(1-\theta)^2}{4\left(\frac{E}{\bar{V}_x L}\right)\theta}\right] d\theta \qquad (4\text{-}105)$$

A family of $\underline{E}(\theta)$ and $\underline{F}(\theta)$ curves, which have been numerically computed and shown in Figures (14) and (15), are plotted with $E/\bar{V}_x L$ as parameters. The correlations among $E/\bar{V}_x L$, the dimensionless parameter $\delta^2\bar{V}_x^2/D$ (or $R^2\bar{V}_x^2/D$) and different flow behavior indexes are shown in Figures (16) and (17).

When $E/\bar{V}_x L$ is very small, the value of $\underline{E}(\theta)$ are essentially zero except for the values of θ close to 1.0. For this small $E/\bar{V}_x L$, the residence time

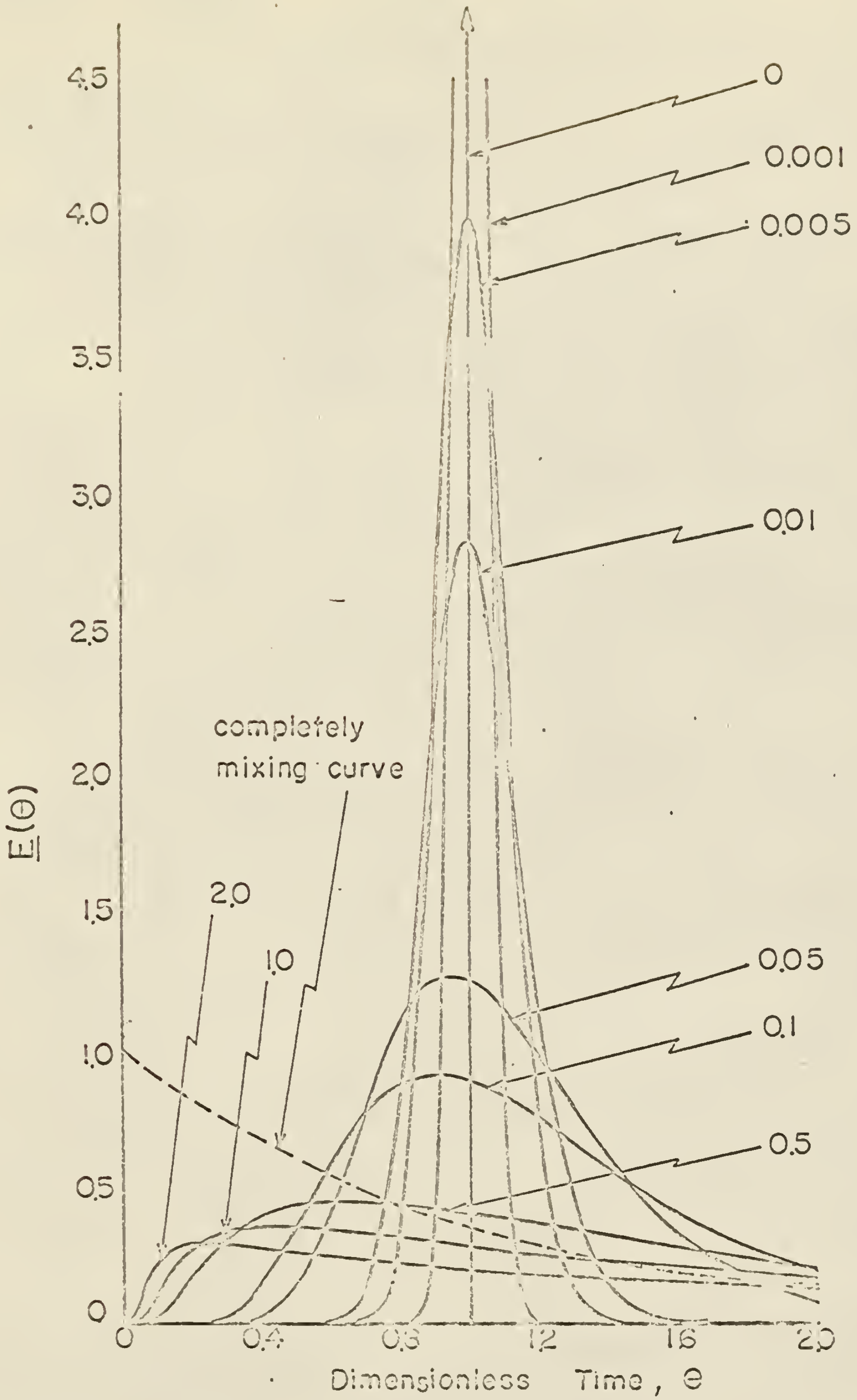

Fig. 14. Generalized $\underline{E}$-curves with $E/\overline{V}_x L$ as the parameter.

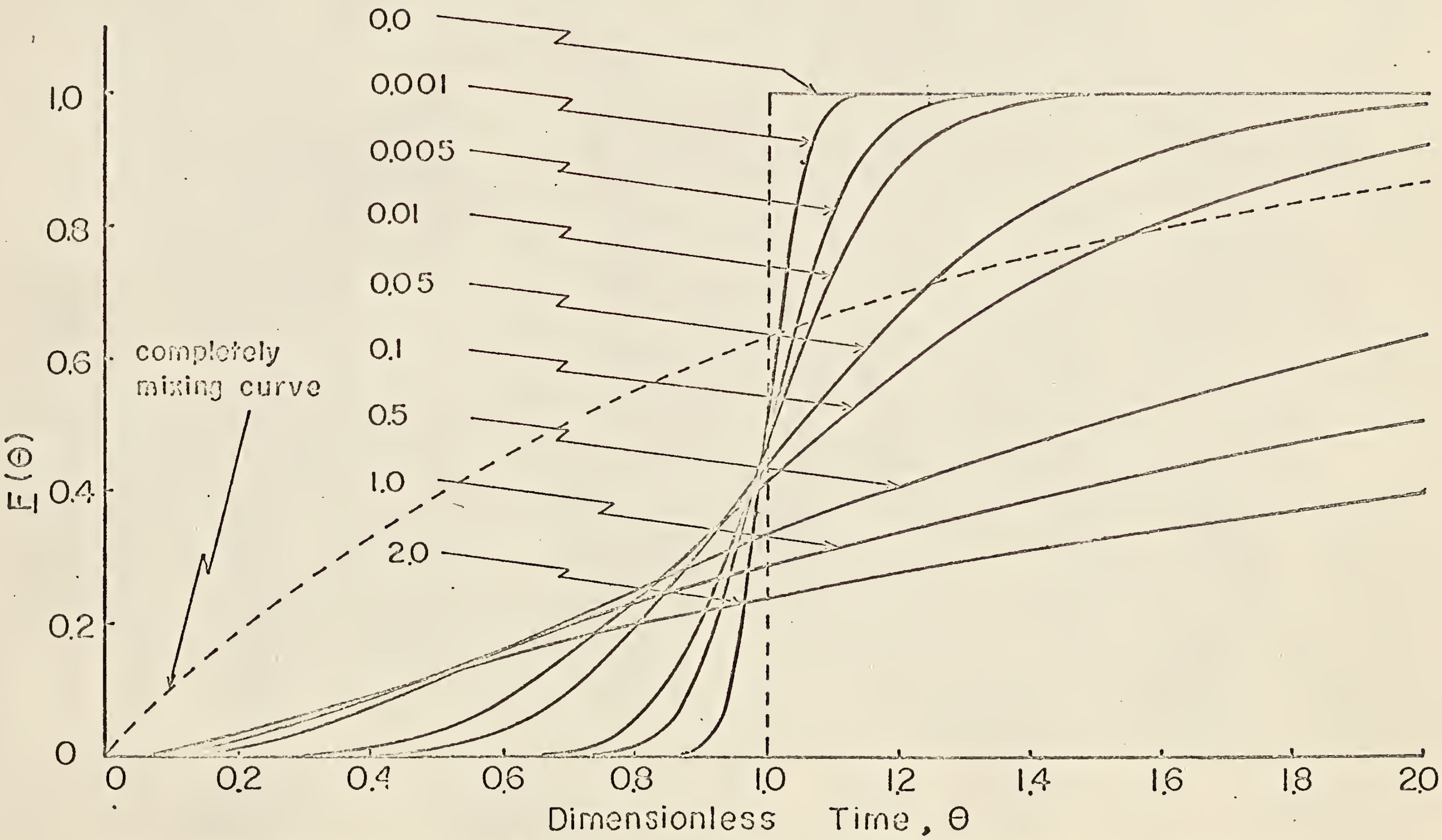

Fig. 15. Generalized $\underline{F}$-curves with $E/\overline{V}_x L$ as the parmeter.

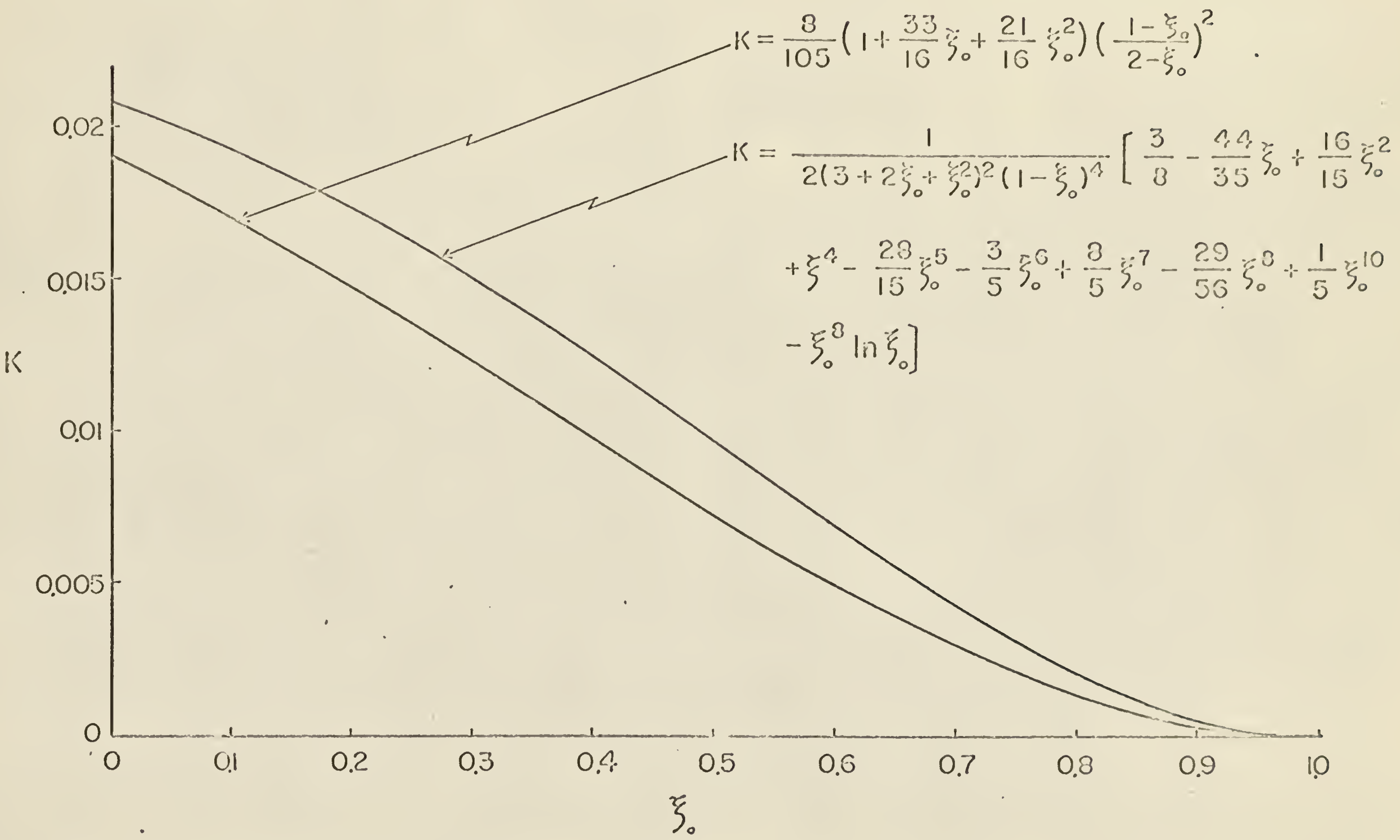

Fig. 16. Taylor dispersion coefficient factors as a function of the Bingham plastic flow behavior index ξ_o.

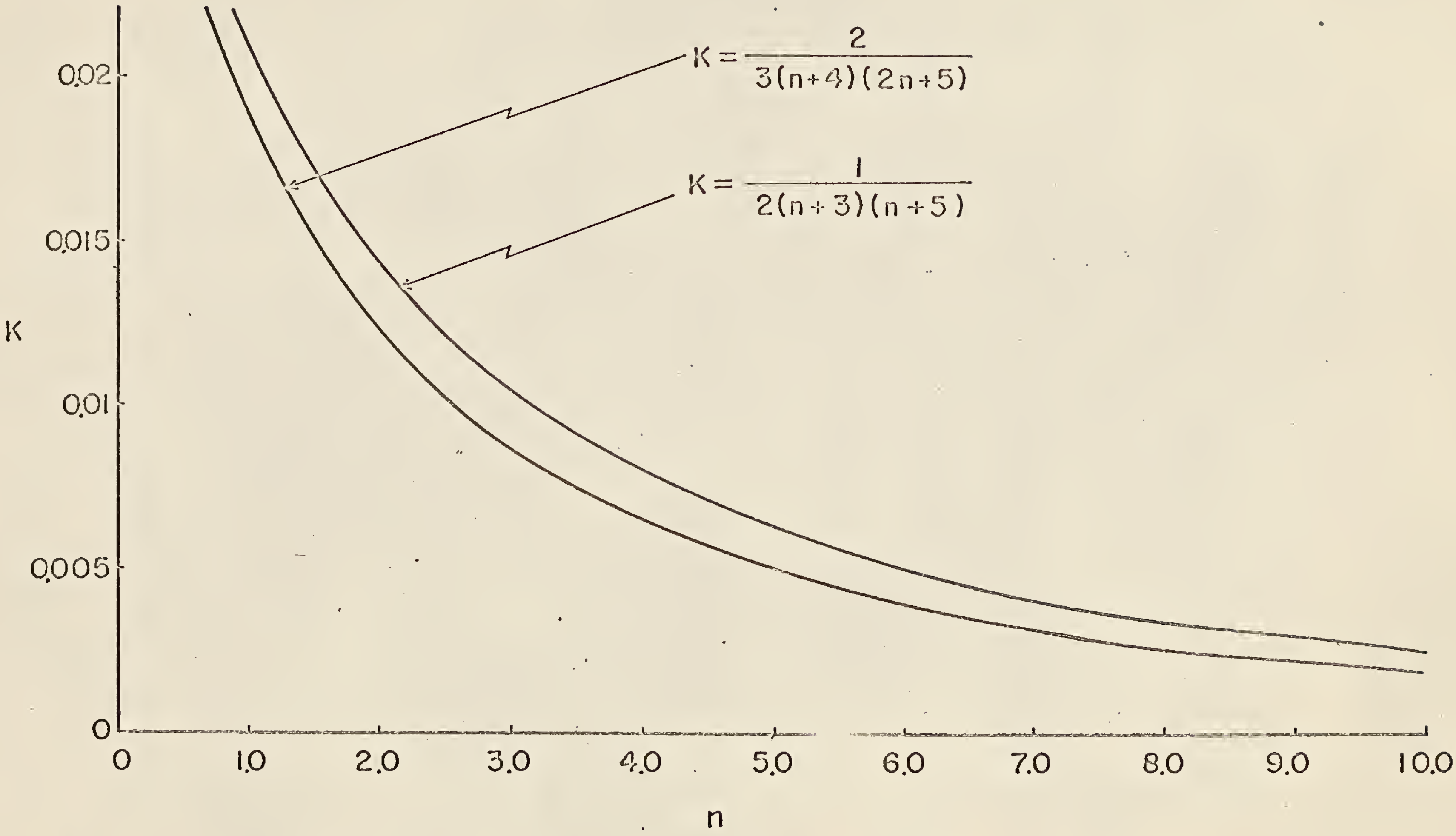

Fig. 17. Taylors dispersion coefficient factors as a function of the power law model flow behavior index n.

distribution functions can approximately be reduced to

$$\underline{E}(\theta) = \frac{1}{2\left[\pi\left(\frac{E}{\bar{V}_x L}\right)\right]^{1/2}} \exp\left[- \frac{(1-\theta)^2}{4\left(\frac{E}{\bar{V}_x L}\right)}\right] \qquad (4\text{-}106)$$

which is a normal distribution function with mean

$$\mu = 1 \qquad (4\text{-}107)$$

and variance

$$\alpha^2 = 2\left(\frac{E}{\bar{V}_x L}\right) \qquad (4\text{-}108)$$

The approximate cumulative age distribution can therefore be expressed by (6)

$$\begin{aligned} \underline{F}(\theta) &= \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left[\frac{1}{2}(1-\theta)(E/\bar{V}_x L)^{\frac{1}{2}}\right] \qquad \text{for} \quad \theta \leq 1 \\ &= \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left[\frac{1}{2}(\theta-1)(E/\bar{V}_x L)^{\frac{1}{2}}\right] \qquad \text{for} \quad \theta \geq 1 \end{aligned} \qquad (4\text{-}109)$$

where

$$\operatorname{erf} z = 2\pi^{-\frac{1}{2}} \int_0^z e^{-z^2}\, dz$$

If one differentiates Equation (104) with respect to θ and sets the derivative equal to zero, the solution shows that the maximum value of $\underline{E}(\theta)$ occurs at time θ_m, that is,

$$\frac{\partial}{\partial\theta}\underline{E}(\theta) = 0 \,, \qquad \text{at} \qquad \theta = \theta_m \qquad (4\text{-}110)$$

or

$$\theta_m^2 + 2\left(\frac{E}{\bar{V}_x L}\right)\theta_m - 1 = 0$$

Therefore

$$\theta_m = \left[\left(\frac{E}{\bar{V}_x L}\right)^2 + 1 \right]^{\frac{1}{2}} - \left(\frac{E}{\bar{V}_x L}\right) \qquad (4\text{-}111)$$

or

$$\theta_m = \frac{1}{\left[\left(\frac{E}{V_x L}\right)^2 + 1\right]^{1/2} + \left(\frac{E}{V_x L}\right)} \qquad (4\text{-}112)$$

This equation shows that the maximum point is close to 1 when $E/\bar{V}_x L$ is far smaller than one. The value of the residence time distribution at the maximum point is obtained by substituting Equation (111) into Equation (104).

The stimulus-response analysis done by Taylor provides a convenient mathematical technique for finding the age distribution of a tracer passing through closed vessels. The usual method of finding the effective dispersion coefficient is to inject a tracer into the system. The tracer concentration is then measured downstream, and the dispersion coefficient may be found from an analysis of the concentration data. The first and second moments are usually needed to characterize a tracer distribution curve. The first moment about the origin, which locates the center of gravity of the tracer curve with respect to the origin, is usually called the mean of a distribution curve and the second moment about this mean which measures the spread of the curve, is called the variance of a distribution. One can find the functional relationship between the mean and variance of the tracer distribution curve and the effective dispersion coefficient. The formulas used to evaluate the mean, μ, and variance, σ^2, of a distribution are

$$\mu = \int_0^\infty \theta \, \underline{E}(\theta) \, d\theta \qquad (4\text{-}113)$$

and

$$\alpha^2 = \int_0^\infty (\theta-\mu)^2 \underline{E}(\theta)\, d\theta$$

or

$$\alpha^2 = \int_0^\infty \theta^2 \underline{E}(\theta)\, d\theta - \left[\int_0^\infty \theta\, \underline{E}(\theta)\, d\theta\right]^2 \qquad (4\text{-}114)$$

Using a perfect delta-function input, the tracer distribution evaluated at $\eta = 1$ has been found to be

$$C = \underline{E}(\theta)$$

$$= \frac{1}{2\left[\pi\left(\frac{E}{\bar{V}_x L}\right)\theta\right]^{1/2}} \exp\left[-\frac{(1-\theta)^2}{4\left(\frac{E}{\bar{V}_x L}\right)\theta}\right] \qquad (4\text{-}104)$$

From this equation the mean and variance are found to be (22, 23),

$$\mu = 1 + 2\left(\frac{E}{\bar{V}_x L}\right) \qquad (4\text{-}115)$$

$$\alpha^2 = 2\left(\frac{E}{\bar{V}_x L}\right) + 8\left(\frac{E}{\bar{V}_x L}\right)^2 \qquad (4\text{-}116)$$

Solving for $E/\bar{V}_x L$ gives

$$\frac{E}{\bar{V}_x L} = \frac{1}{4}\left[(8\alpha^2 + 1)^{\frac{1}{2}} - 1\right] \qquad (4\text{-}117)$$

If the velocity distribution of the fluid passing through the system is known, the dispersion coefficient can be obtained by injecting an impulse of tracer into the system and finding the variance from the experimental data.

When $E/\bar{V}_x L$ is far smaller than 1, Equations (115) and (116) approximate

$$\mu = 1$$

$$\alpha^2 = 2(\frac{E}{\bar{V}_x L})$$

which are the same as Equations (107) and (108).

Consider a system of steady state flow in a straight pipe from the point of view of their dimensions. It has been found (26) that the dimensionless group $E/\bar{V}_x d$ is a function of the Reynolds number, the Schmidt number and a relative roughness number. The roughness factor has been shown to be important only in turbulent flow. The functional relationship for laminar flow in a pipe may therefore be expressed as

$$\frac{E}{\bar{V}_x d} = \phi\left[(\frac{d\bar{V}_x \rho}{\mu}), (\frac{\mu}{\rho D})\right] \qquad (4\text{-}118)$$

The generalized expression of the effective dispersion coefficient is

$$E = \frac{K}{4} \frac{d^2 \bar{V}_x^2}{D} \qquad (4\text{-}119)$$

or

$$\frac{E}{\bar{V}_x d} = \frac{K}{4} (\frac{d\bar{V}_x \rho}{\mu})(\frac{\mu}{\rho D}) \qquad (4\text{-}120)$$

This relation is applicable only when the system satisfies Taylor's limiting conditions, that is,

$$\frac{L}{V_m} >> 0.0682 \frac{R^2}{D}$$

for fluid flow in a cylindrical tube and

$$\frac{L}{V_m} >> 0.101 \frac{\delta^2}{D}$$

for fluid flow through a slit. A family of curves with $E/\bar{V}_x d$ as a function of Reynolds number and Schmidt number as the parameters is shown in Figure (18). Equation (119) can also be written as

$$\frac{E}{\bar{V}_x d} = \frac{K}{4}\left(\frac{d\bar{V}_x}{D}\right) \qquad (4\text{-}121)$$

From this a family of curves of slope 1 and intercept K/4 is resulted as plotted in Figures (19) through (21).

DISPERSION OF MATTER IN TURBULENT FLOW OF NON-NEWTONIAN FLUIDS.

In a later paper Taylor (7) has shown that the analogous problem of dispersion in turbulent flow can be solved in the same way as that used for laminar flow. In the case of Newtonian flow, the effective dispersion coefficient E is found to be 10.1 RV_* or $E = 7.14\ R\bar{V}_x/\sqrt{f}$ (26) in which f is the Fanning friction factor and V_* the friction velocity. The effective dispersion coefficient is almost entirely due to the variation of the time-average velocity with radius and the effect of turbulent axial diffusion is almost negligible in this connection. Taylor's calculation assumed the validity of Reynolds' analogy, and the universal velocity distribution. The turbulent flow region is of the greatest practical importance in the fact that most fluid flow in industrial systems is turbulent. It is the purpose of this section to extend the analysis to non-Newtonian turbulent flow systems. The theoretical analysis for turbulent flow of non-Newtonian fluids through smooth round tubes was performed for the first time by Dodge and Metzner (3). By analysis and experiments, they showed that a purely viscous non-Newtonian fluid provides a small amount of drag reduction. Various observations and conclusions regarding the different trends in frictional drag were made by many investigators. However, the turbulent velocity profile suggested by

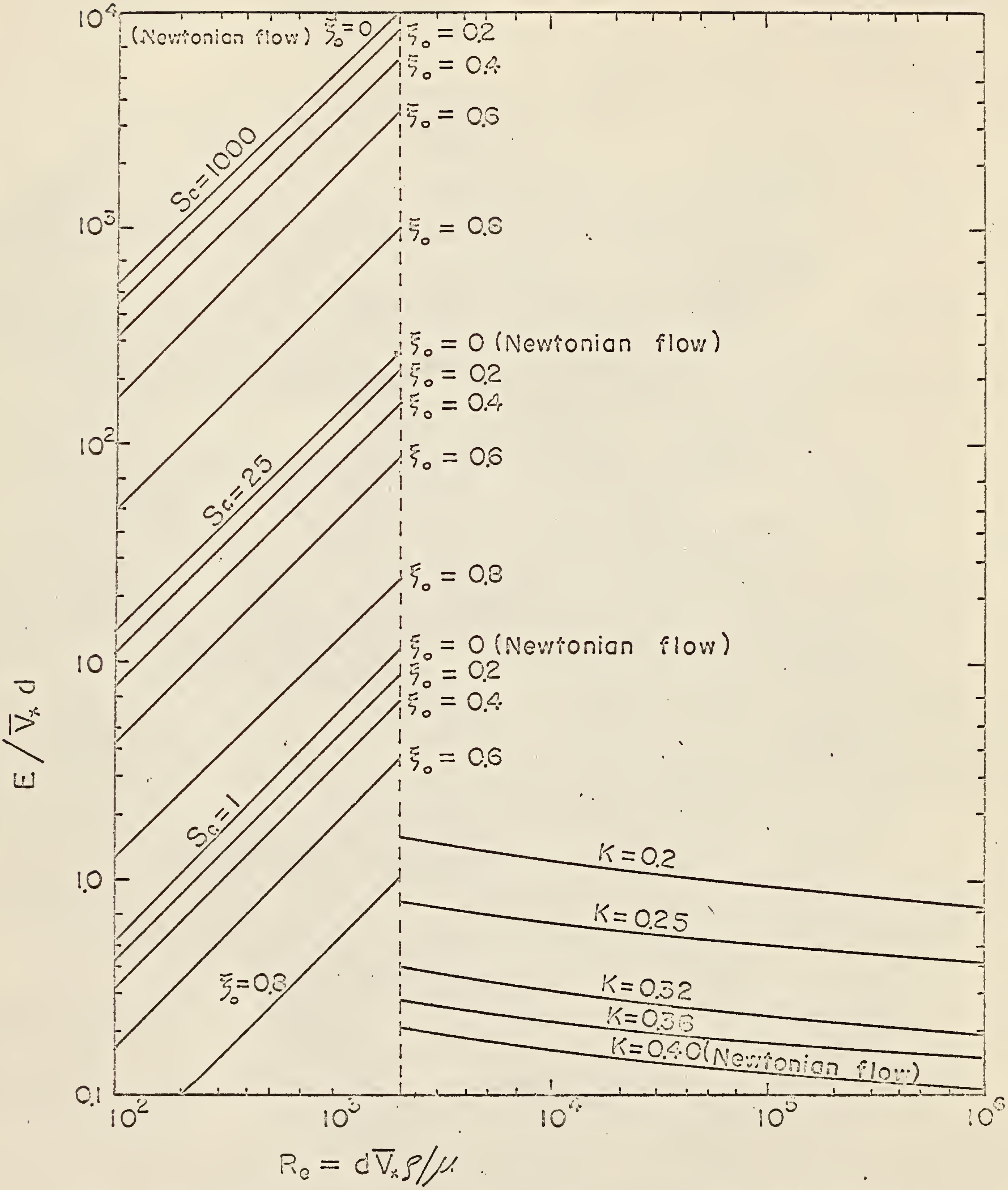

Fig. 18. Correlation for the dispersion of fluids flowing in pipes.

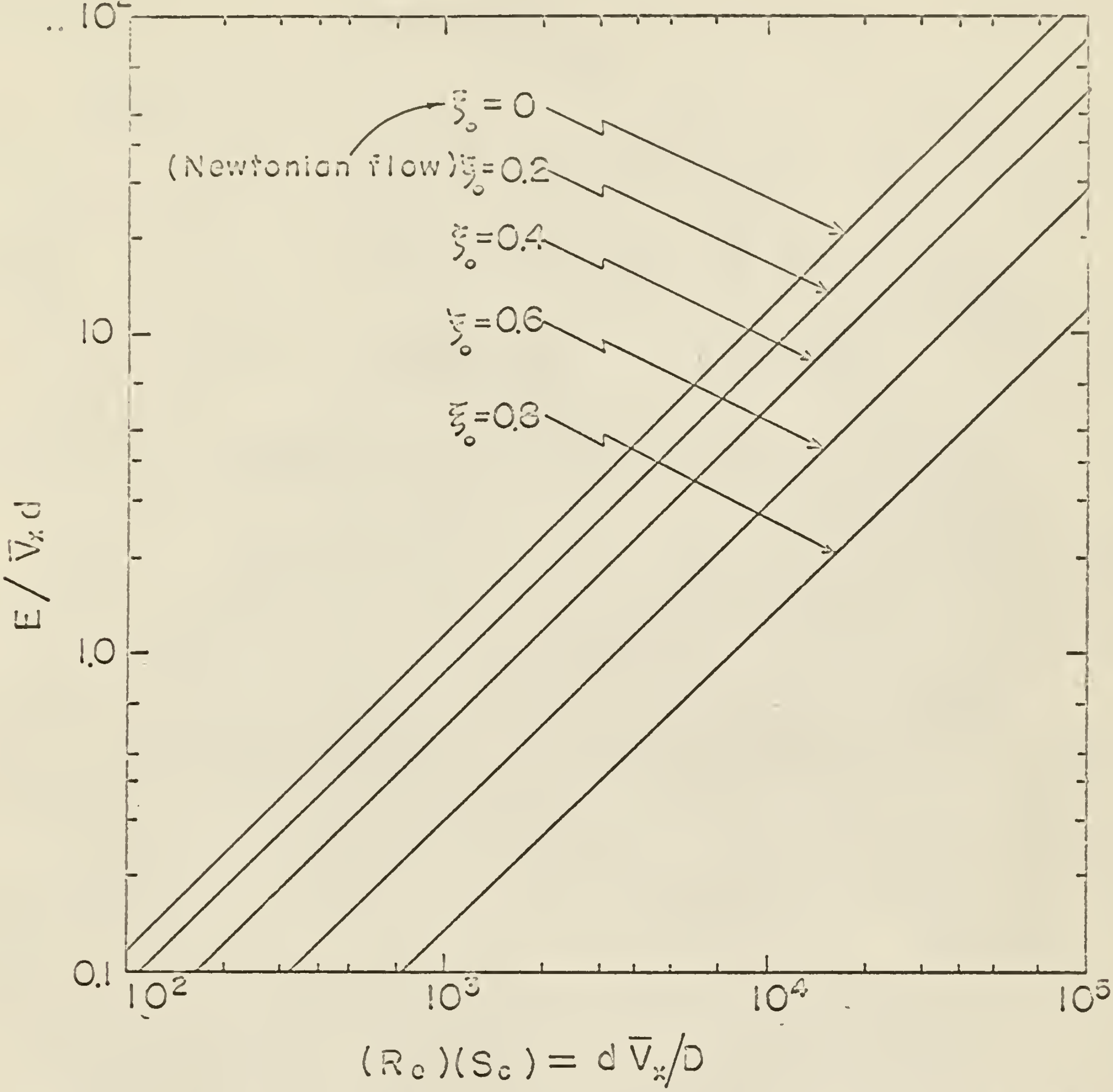

Fig. 19. Correlation of dispersion coefficient of Bingham plastic fluid in lamina flow in a slit.

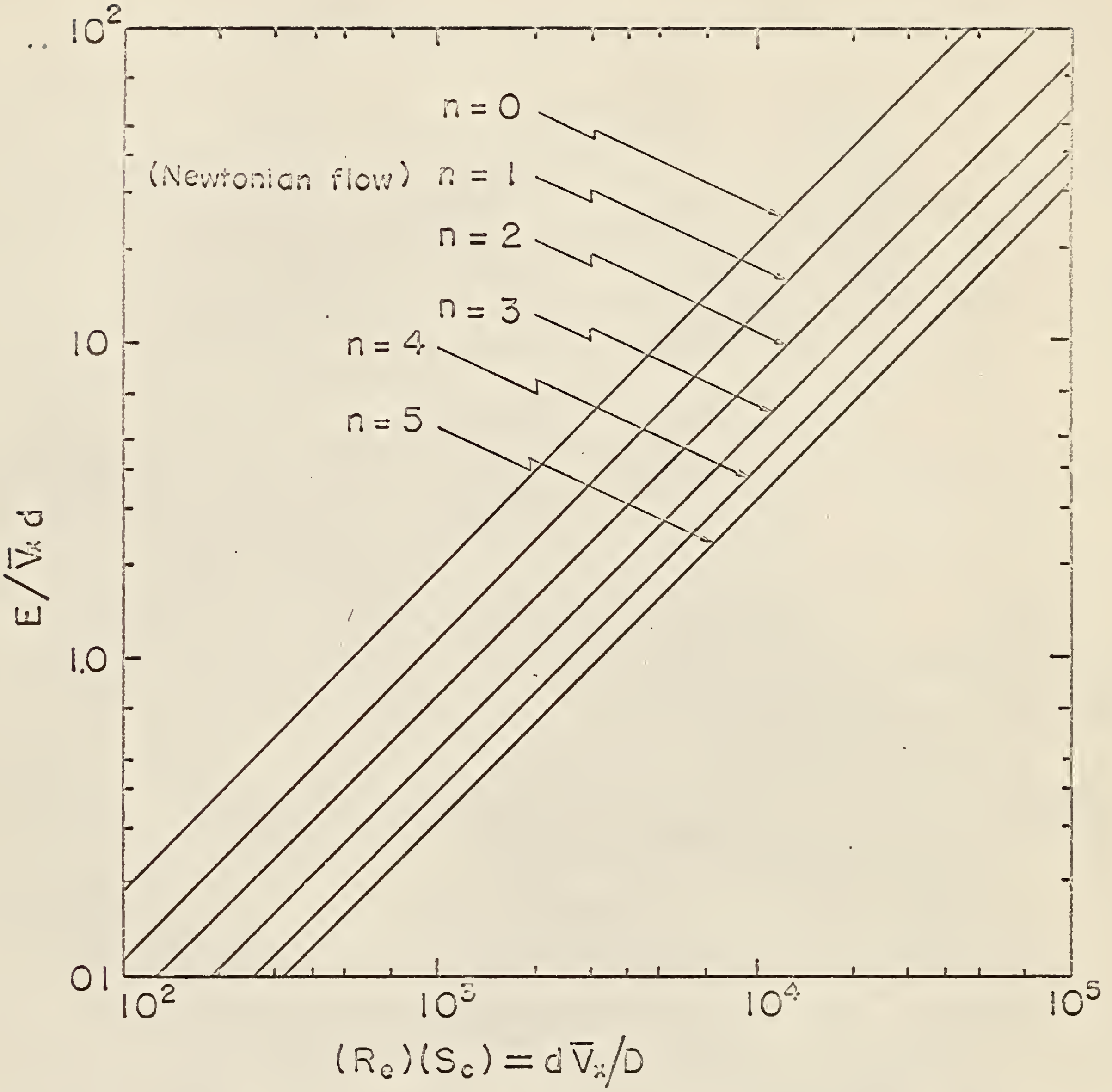

Fig. 20. Correlation of dispersion coefficiet of Ostwald-de Waele fluids in laminar flow in a slit.

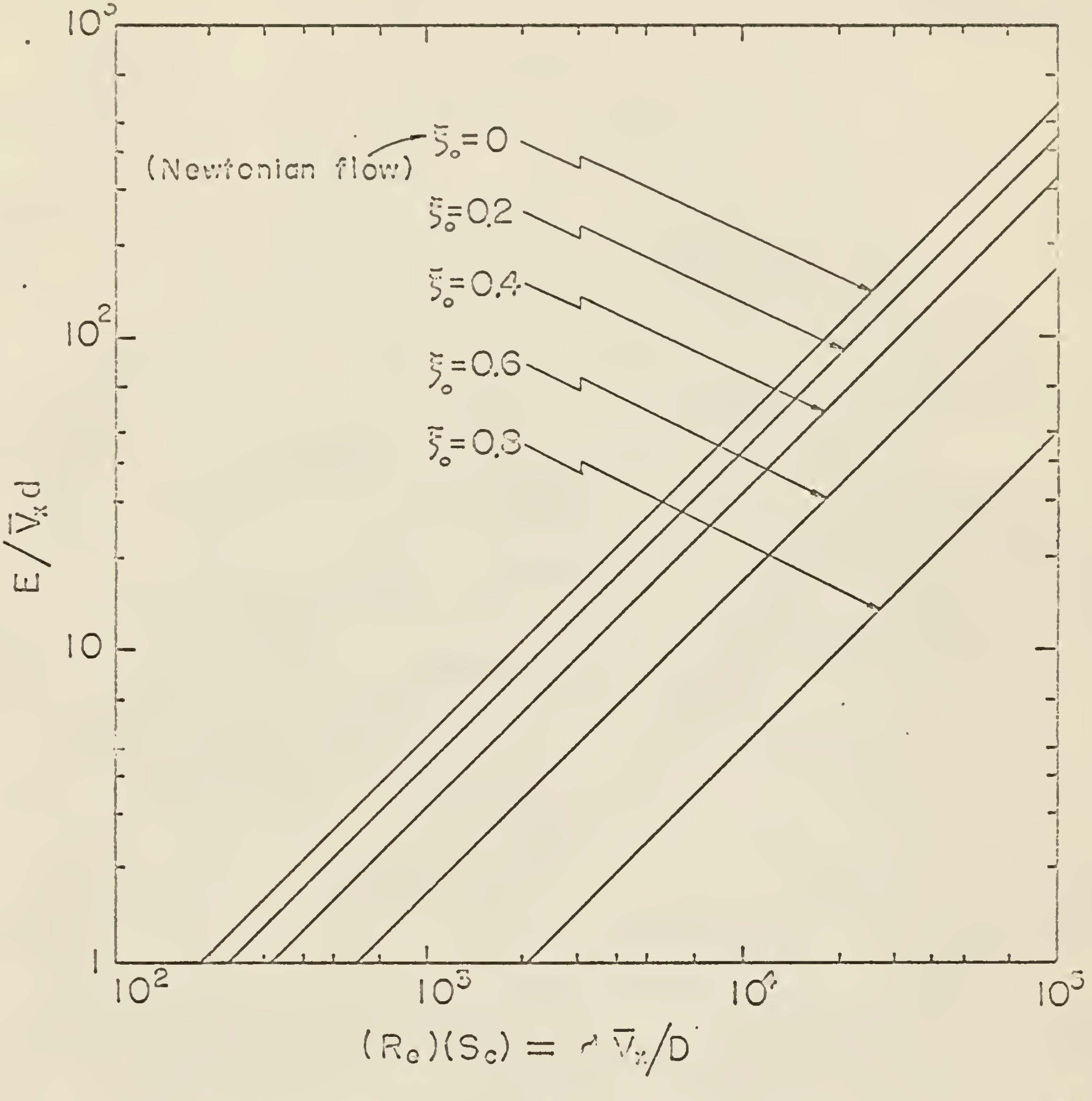

Fig. 21. Correlation of dispersion coefficinent of Bingham plastic fluids in laminar flow in a cylindrical tube.

Wells (24) is assumed in the present analysis, that is, the velocity V_x at radius r in a pipe of diameter 2R is given by

$$\frac{V_x}{V_*} = \frac{1}{k} \ln (1-\xi) + C_1 \tag{4-122}$$

where

$\xi = r/R$

k = mixing length constant

V_* = Friction velocity, defined by $(\tau_0/p)^{\frac{1}{2}}$

C_1 = constant of integration

It should be noted that when k = 0.4, Equation (122) reduces to the expression for Newtonian fluids given by Prandtl (25), which is usually known as the "universal velocity profile". Reynolds' analogy will also be assumed to be true, which ensures that the transfer of matter, heat and momentum by turbulence are exactly analogous. Let ϵ be the coefficient of transfer, Reynolds' analogy in the present case may be expressed as

$$\epsilon = \frac{\tau}{\rho \frac{\partial V_x}{\partial r}} = \frac{m}{\frac{\partial C}{\partial r}} \tag{4-123}$$

in which τ is the shear stress at radius r and m is the rate of radial transfer of matter of concentration C. Two different flow processes, one in which a fluid flows through a slit and the other in which a fluid flows through a circular tube are treated separately.

1. Flow through a slit

The velocity distribution in the slit is

$$\frac{V_x}{V_*} = \frac{1}{k} \ln (1 - \zeta) + C_1 \tag{4-124}$$

where

$$\zeta = |y/\delta|$$

and Reynolds' analogy may be expressed by the equation

$$\epsilon = \frac{\tau}{\rho(\partial V_x/\partial y)} = \frac{\dot{m}}{\partial C/\partial y} \tag{4-125}$$

From Equation (124) the mean velocity $\bar{V}_x$ may be found as

$$\bar{V}_x = V_* \int_0^1 V_x \, d\zeta$$

$$= V_* \int_0^1 \left[\frac{1}{k} \ln (1-\zeta) + C_1 \right] d\zeta$$

$$= V_* \left(C_1 - \frac{1}{k}\right) \tag{4-126}$$

The viscous stress τ at ζ is related to τ_0 by the equation

$$\tau = \tau_0 \zeta^{-1} \tag{4-127}$$

Substituting τ from Equation (127) into Equation (125) yields

$$\epsilon = \frac{\tau_0 \zeta^{-1}}{\rho(\partial V_x/\partial y)} \tag{4-128}$$

From Equations (124) and (128), one has

$$\epsilon = k\delta V_* \zeta (1 - \zeta) \tag{4-129}$$

Based on Reynolds' analogy the equation for conservation of C is

$$\frac{\partial C}{\partial y} \left(\epsilon \frac{\partial C}{\partial y}\right) = V_x \frac{\partial C}{\partial x} + \frac{\partial C}{\partial t} \tag{4-130}$$

Introducing the dimensionless variable $\zeta = |y/\delta|$ and substituting for ϵ and V_x from Equations (129) and (124), Equation (130) becomes

$$\frac{\partial}{\partial \zeta}\left[k\zeta(1-\zeta)\,\frac{\partial C}{\partial \zeta}\right]=\left[\frac{1}{k}\ln\,(1-\zeta)+C_1\right]\delta\,\frac{\partial C}{\partial x}+\frac{\delta}{V_*}\left(\frac{\partial C}{\partial t}\right) \qquad (4\text{-}131)$$

where x is measured along the slit from a fixed point. It is convenient in the present discussion to define concentration and velocity relative to the axes which move with the mean flow. Relative to these axes, the velocity is

$$V_{x1} = V_x - \bar{V}_x$$

$$= V_*\left[\frac{1}{k}\ln\,(1-\zeta)+\frac{1}{k}\right] \qquad (4\text{-}132)$$

and Equation (131) becomes

$$\frac{\partial}{\partial \zeta}\left[k\zeta(1-\zeta)\,\frac{\partial C}{\partial \zeta}\right]=\frac{1}{k}\left[\ln\,(1-\zeta)+1\right]\delta\,\frac{\partial C}{\partial x_1}+\frac{\delta}{V_*}\,\frac{\partial C}{\partial t_1} \qquad (4\text{-}133)$$

The small radial variation in C can therefore be calculated from the equation

$$\frac{\partial}{\partial \zeta}\left[k^2\zeta(1-\zeta)\,\frac{\partial C}{\partial \zeta}\right]=\left[\ln\,(1-\zeta)+1\right]\delta\,\frac{\partial C}{\partial x_1} \qquad (4\text{-}134)$$

where $\partial C/\partial x_1$ is considered to be independent of ζ. C is then of the form

$$C = C_{x_1} + C_\zeta \qquad (4\text{-}135)$$

where C_{x_1} is independent of ζ but varies linearly with x_1, and C_ζ is independent of x_1. Equation (134) can then be solved as

$$C_\zeta = \frac{\delta}{k^2}\,\frac{dC_{x1}}{dx_1}\int_0^\zeta \frac{1}{\zeta(1-\zeta)}\int_0^\zeta\left[\ln\,(1-\zeta)+1\right]d\zeta d\zeta$$

$$= \frac{\delta}{k^2}\,\frac{dC_{x1}}{dx_1}\int_0^\zeta -\frac{1}{\zeta}\ln\,(1-\zeta)\,d\zeta$$

$$= \frac{\delta}{k^2}\,\frac{dC_{x1}}{dx_1}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\zeta^n\right) \qquad (4\text{-}136)$$

The rate of transfer of C across the section at x_1 is

$$Q = 2W\delta \int_0^1 C_\zeta V_{x1}\, d\zeta$$

$$= 2W\delta \cdot \frac{\delta V_*}{k^2} \left(\frac{dC_{x1}}{dx_1}\right) \int_0^1 \left[\frac{1}{k} \ln(1-\zeta) + \frac{1}{k} \right] \sum_{n=1}^{\infty} \left(\frac{1}{n^2} \zeta^n\right) d\zeta$$

$$= 2W\delta \frac{\delta V_*}{k^3} \left(\frac{dC_{x1}}{dx_1}\right) \int_0^1 \left[\ln(1-\zeta) \sum_{n=1}^{\infty} \left(\frac{1}{n^2} \zeta^n\right) + \sum_{n=1}^{\infty} \left(\frac{1}{n^2} \zeta^n\right) \right] d\zeta$$

$$= -0.808\, W\delta \frac{V_*}{k^3} \left(\frac{dC_{x1}}{dx_1}\right) \qquad (4\text{-}137)$$

The resulting expression shows that an effective coefficient of diffusion E would transfer matter across a section at rate $-2W\delta E(dC_{x1}/dx_1)$, so that matter is transferred relative to a plane which moves with its mean velocity exactly as though it were dispersed by an apparent coefficient of diffusion

$$E = 0.404 \frac{\delta V_*}{k^3} \qquad (4\text{-}138)$$

In order to estimate the effect of longitudinal diffusion in turbulent diffusion, it is probably not unreasonable to assume that the coefficient of longitudinal diffusion is equal to ϵ. The rate of transfer of matter across a plane due to longitudinal diffusion is

$$Q' = 2W\delta \frac{dC}{dx} \int_0^1 \epsilon\, d\zeta \qquad (4\text{-}139)$$

Since

$$\epsilon = k \delta V_* \zeta (1-\zeta) \qquad (4\text{-}129)$$

one has

$$Q' = 0.333\, W\delta^2\, kV_* \left(\frac{dC}{dx}\right), \qquad (4\text{-}140)$$

which implies that the mean coefficient of diffusion due to the longitudinal components of turbulent velocity is

$$E' = 0.167\, k\delta V_{*} \tag{4-141}$$

Therefore, the corrected value of E which including the longitudinal turbulent diffusion is obtained by adding Equations (138) and (141), that is,

$$E = \left(\frac{0.404}{k^3} + 0.167\, k\right) \delta V_{*} \tag{4-142}$$

By definition

$$\frac{V_{*}}{\bar{V}_x} = \frac{1}{\bar{V}_x} \sqrt{\tau_0/\rho} \tag{4-143}$$

while the Fanning friction factor f is defined by

$$\tau_0 = \frac{1}{2} f \rho \bar{V}_x^2 \tag{4-144}$$

Thus, the friction velocity V_{*} and the Fanning friction factor f can be correlated by (26)

$$\frac{V_{*}}{\bar{V}_x} = \sqrt{\frac{1}{2} f} \tag{4-145}$$

Therefore, one obtains

$$E = \left(\frac{0.404}{k^3} + 0.167\, k\right) \delta \bar{V}_x \sqrt{\frac{1}{2} f} \tag{4-146}$$

$$\frac{E}{4\delta \bar{V}_x} = \frac{\sqrt{2}}{8} \left(\frac{0.404}{k^3} + 0.167\, k\right) \sqrt{f} \tag{4-147}$$

When k = 0.4, that is, for the special case of the Newtonian system (2) the final expressions are

$$E = 6.48\,\delta\bar{V}_x \tag{4-148}$$

$$\frac{E}{4\delta\bar{V}_x} = 1.14\,\sqrt{f} \tag{4-149}$$

2. Flow through Cylindrical Tubes

The generalized velocity distribution and Reynolds' analogy have been given as (7)

$$\frac{V_x}{V_*} = \frac{1}{k}\ln\,(1-\xi) + C_1 \tag{4-122}$$

where

$$\xi = r/R$$

and

$$\epsilon = \frac{\tau_0\,\xi}{\rho\,\partial V_x/\partial r} \quad \frac{m}{\partial C/\partial y} \tag{4-123}$$

From Equations (122) and (123) we have

$$\epsilon = kRV_*\,\xi(1-\xi) \tag{4-150}$$

The mean velocity may also be found as

$$\bar{V}_x = 2V_*\int_0^1\left[\frac{1}{k}\ln\,(1-\xi) + C_1\right]\xi\,d\xi$$

$$= V_*\,(C_1 - \frac{3}{2k}) \tag{4-151}$$

By using Reynolds' analogy, the equation for conservation of C can be written in cylindrical coordinates as

$$\frac{\partial C}{\partial r}\,(\epsilon r\,\frac{\partial C}{\partial r}) = r(V_x\,\frac{\partial C}{\partial x} + \frac{\partial C}{\partial t}) \tag{4-152}$$

Introducing dimensionless variables and substituting for ϵ and V_x from Equations (150) and (122), Equation (152) becomes

$$\frac{\partial}{\partial \xi}\left[k\xi^2(1-\xi)\frac{\partial C}{\partial \xi}\right] = \xi\left\{\left[\frac{1}{k}\ln(1-\xi) + C_1\right]R\frac{\partial C}{\partial x} + \frac{R}{V_*}\frac{\partial C}{\partial t}\right\} \qquad (4\text{-}153)$$

It is convenient to use axes which move with the mean speed of flow, which is defined by

$$x_1 = x - \bar{V}_x t \qquad (4\text{-}154)$$

Relative to these axes, the velocity distribution is

$$V_{x1} = V_x - \bar{V}_x = \frac{V_*}{k}\left[\ln(1-\xi) + \frac{3}{2}\right] \qquad (4\text{-}155)$$

and the dispersion equation becomes

$$\frac{\partial}{\partial \xi}\left[k\xi^2(1-\xi)\frac{\partial C}{\partial \xi}\right] = \xi\left\{\frac{1}{k}\left[\ln(1-\xi) + \frac{3}{2}\right]R\frac{\partial C}{\partial x_1} + \frac{R}{V_*}\frac{\partial C}{\partial t_1}\right\} \qquad (4\text{-}156)$$

The radial variation in C can be calculated by setting $\partial C/\partial t_1 = 0$ and $\partial C/\partial x_1$ being independent of x_1 and ξ. C is then of the form

$$C = C_{x1} + C_\xi \qquad (4\text{-}157)$$

where C_{x1} is independent of ξ but varies linearly with x_1, and C_ξ is independent of x_1. Equations (156) is then solved as

$$C_\xi = \frac{R}{k^2}\frac{dC_{x1}}{dx_1}\int_0^\xi \frac{1}{\xi^2(1-\xi)}\int_0^1\left[\ln(1-\xi) + \frac{3}{2}\right]\xi\, d\xi\, d\xi$$

$$= \frac{R}{2k^2}\frac{dC_{x1}}{dx_1}\int_0^\xi\left[\frac{1}{\xi^2}\ln(1-\xi) + \frac{1}{\xi}\ln(1-\xi) + \frac{1}{\xi}\right]d$$

$$= -\frac{R}{2k^2}\frac{dC_{x1}}{dx_1}\left[\ln(1-\xi) - \frac{1}{\xi}\ln(1-\xi) - 1 - \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\xi^n\right)\right] \qquad (4\text{-}158)$$

The rate of transfer of C across the section at x_1 is then

$$Q = \pi R^2 V_* \frac{R}{k^2} \frac{dC_{x1}}{dx_1} \int_0^1 \left[\ln (1-\xi) + \frac{3}{2} \right] \left[\ln (1-\xi) - \frac{1}{\xi} \ln (1-\xi) - 1 - \sum_{n=1}^{\infty} \left(\frac{1}{n^2} \xi \right) \right] \xi \, d\xi$$

$$= 0.328 \, \pi R^3 V_* \frac{1}{k^3} \frac{dC_{x1}}{dx_1} \qquad (4\text{-}159)$$

The effect of the longitudinal component of turbulent diffusion can be estimated by

$$Q' = - 2\pi \left(\frac{dC}{dx}\right) \int_0^R \epsilon \, rdr$$

$$= - 2\pi R^3 V_* k \left(\frac{dC}{dx}\right) \int_0^1 \xi^2 (1-\xi) \, d\xi$$

$$= - 0.167 \, \pi R^3 k V_* \frac{dC}{dx} \qquad (4\text{-}160)$$

Q and Q' are additive. Comparing Equations (159) and (160) with $Q = \pi R^2 E \frac{dC_{x1}}{dx_1}$, the corrected value of E allowing for longitudinal diffusion is

$$E = \left(\frac{0.328}{k^3} + 0.167 \; k\right) R V_* \qquad (4\text{-}161)$$

Since the friction velocity V_* and the Fanning friction factor f are correlated by

$$V_* = \bar{V}_x \sqrt{\frac{1}{2} f} \qquad (4\text{-}145)$$

one obtains

$$E = \frac{\sqrt{2}}{2} \left(\frac{0.328}{k^3} + 0.167 \; k\right) R \bar{V}_x \sqrt{f} \qquad (4\text{-}162)$$

and

$$\frac{E}{\bar{V}_x D} = \frac{\sqrt{2}}{4} \left(\frac{0.328}{k^3} + 0.167\ k\right) \sqrt{f} \qquad (4\text{-}163)$$

For the special case of the Newtonian flow, these expressions reduce to

$$E = 5.19\ RV_* \qquad (4\text{-}164)$$

and

$$\frac{E}{\bar{V}_x d} = 1.84 \sqrt{f} \qquad (4\text{-}165)$$

Comparing Equation (165) with Taylor's result, $E/\bar{V}_x d = 3.57 \sqrt{f}$, it is seen that a large difference exists. But if one sets $k = 0.32$, one can obtain from Equations (161) and (163)

$$E = 10.054\ (RV_*) \qquad (4\text{-}166)$$

and

$$\frac{E}{\bar{V}_x d} = 3.56 \sqrt{f} \qquad (4\text{-}167)$$

which are identical to Taylor's results. The velocity distribution for $k = 0.32$, $k = 0.4$ (generally accepted for turbulent Newtonian flow) and the experimentally measured velocity profile used by Taylor are plotted on Figure (22) for comparison. One conclusion which may be drawn is that the magnitude of the turbulent dispersion coefficient depends mainly on the velocity distribution near the tube wall. As shown in Figure (22) the velocity profile used by Taylor and the universal velocity profile for $k = 0.32$ coincide for ξ in the interval $0.9 < \xi < 1$. One can also see from the figure that a large difference exists between Taylor's experimentally measured velocity distribution and the theoretical result based on Prandtl's

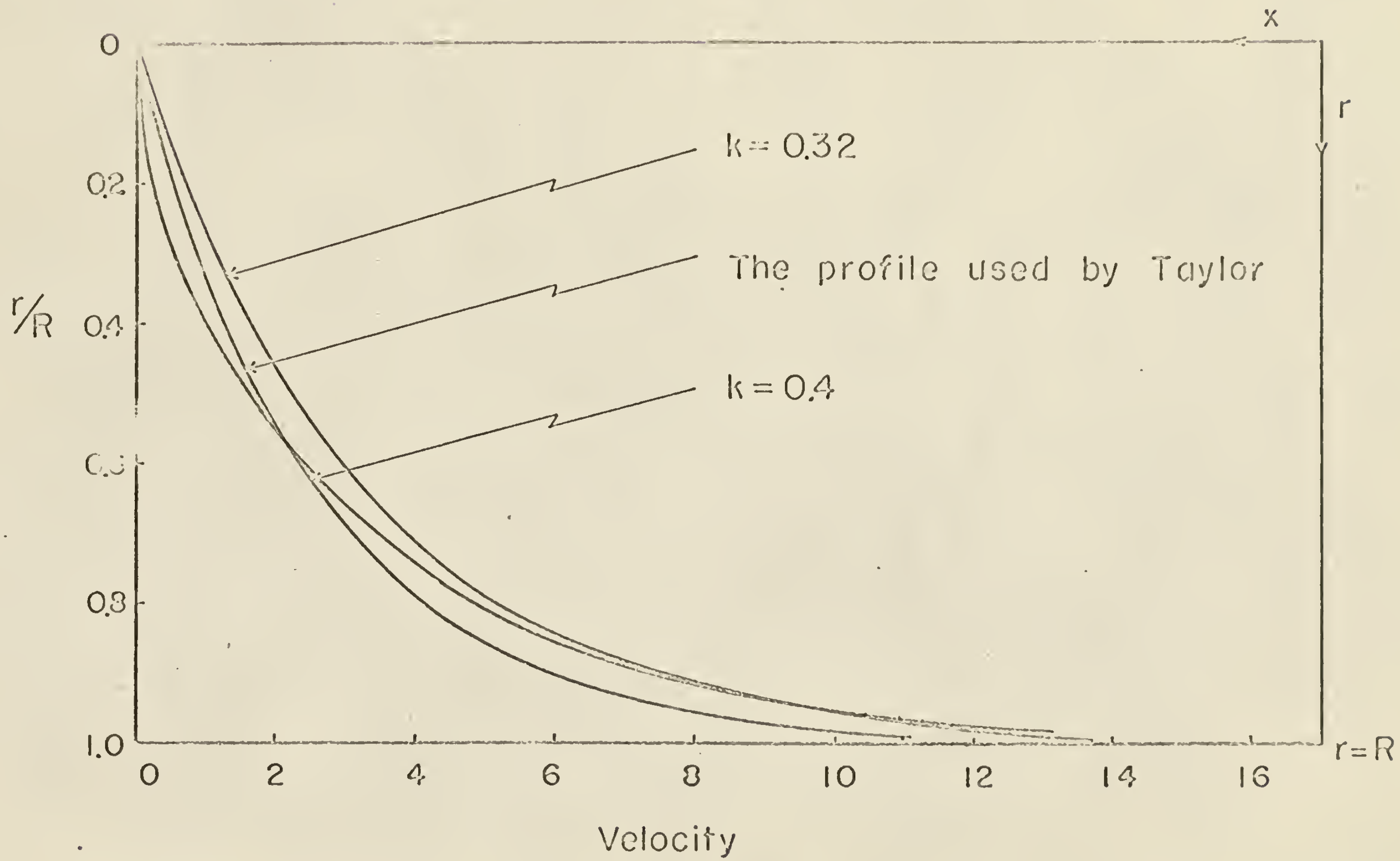

Fig. 22. Velocity distribution in a turbulent flow system.

mixing length theory (25). From the stand point of fluid dispersion, however, the universal velocity profile with $k = 0.32$ is more appropriate in interpreting the result than the generally accepted value of $k = 0.4$.

Levenspiel (26) has mentioned that the molecular diffusion in turbulent flow does not contribute significantly to the transport of material. The molecular diffusion is greatly overshadowed by turbulent eddy mixing. Thus, a dimensionless group involving the longitudinal dispersion coefficient may be expressed in terms of the Reynolds' number and the relative roughness factor, that is,

$$\frac{E}{\bar{V}_x d} = \phi\left[\frac{d\bar{V}_x \rho}{\mu}, \frac{\epsilon}{d}\right] \tag{4-168}$$

In a theoretical analysis of mixing in turbulent flow, $E/\bar{V}_x d$ has been shown to be a function of the Fanning friction which is based on the assumption that Reynolds' analogy holds. In turn, however, f has been experimentally found to be a function of the Reynolds number and the relative roughness parameter. Thus, one can see that Equation (166) is valid. A family of curves which correlate $E/\bar{V}_x d$, the longitudinal dispersion number, and the Reynolds number are shown in Figure (18). The curves for smooth tubes only and thus the roughness factor is not indicated in the figure.

V. EXTENSION OF ARIS' WORK TO NON-NEWTONIAN FLUID FLOW

If a pulse of a tracer is injected into a flowing stream, this discontinuity spreads out as it moves with the fluid past a downstream measuring point. Levenspiel and Smith (27) have shown that the variance of a tracer curve can conveniently relate this spread to the dispersion coefficient. Taylor's analysis has been proved to be an easy way to relate the variance of the tracer distribution and the effective dispersion coefficient. Since the partial differential equation for diffusion cannot be solved in general, Aris (9) presented a new approach to the analysis of the problem, which calculates the moments of the distribution in terms of their respective parameters, and then compares the moments with the moments of the axial-dispersed slug flow model to find the relationship between parameters. His analysis removed the restrictions Taylor imposed on some of the parameters. The result shows that the rate of growth of the variance is proportional to the sum of the molecular diffusion coefficient D and the effective dispersion coefficient suggested by Taylor. It is the purpose of this chapter to extend Aris' analysis to the Ostwald-de Waele fluid.

The equation of continuity of the tracer in a system with constant density ρ and molecular diffusion coefficient D is generally expressed as

$$\frac{\partial C}{\partial t} = D \nabla^2 C - V \cdot \nabla C$$

It is convenient to define concentration and velocity relative to axes which move with the mean speed of flow and to write the equation of continuity in dimensionless form. Consider a tube of circular cross section and cylindrical coordinates within the tube (see Fig. (2)). Let

$$\eta = (x - \bar{V}_x t)/L, \qquad \xi = r/R, \qquad \theta = t\bar{V}_x/L$$

The governing diffusion equation becomes

$$\frac{\partial C}{\partial \theta} = \frac{LD}{R^2\bar{V}_x}\left[\frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial C}{\partial \xi}\right) + \frac{1}{\xi^2}\frac{\partial^2 C}{\partial \phi^2}\right] + \frac{D}{L\bar{V}_x}\frac{\partial^2 C}{\partial \eta^2} - \left(\frac{2}{n+1} - \frac{n+3}{n+1}\xi^{n+1}\right)\frac{\partial C}{\partial \eta} \tag{5-2}$$

which is to be solved subject to the initial and boundary conditions

$$c(\eta, \xi, \phi, 0) = c_0(\eta, \xi, \phi), \tag{5-3}$$

$$\frac{\partial C}{\partial \xi} = 0 , \qquad \text{at } \xi = 1 \tag{5-4}$$

Let

$$c_p(\xi, \phi, \theta) = \int_{-\infty}^{\infty} \eta^p c(\eta, \xi, \phi, \theta)\, d\eta \tag{5-5}$$

and

$$M_p(\theta) = \bar{c}_p = \frac{1}{\pi R^2}\int_0^{2\pi} d\phi \int_0^1 c_p(\xi, \phi, \theta)\,\xi\, d\xi \tag{5-6}$$

be the pth moment of the distribution of the solute as a function of ξ, ϕ and θ and the pth moment of the distribution of the solute in the tube. Multiplying Equation (2) by η^p and integrating the resulting equation with respect to η from $-\infty$ to ∞, which is performed by integration by parts, yield

$$\frac{\partial C_p}{\partial \theta} = \frac{LD}{R^2\bar{V}_x}\left[\frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial C_p}{\partial \xi}\right) + \frac{1}{\xi^2}\frac{\partial^2 C_p}{\partial \phi^2}\right] + p(p-1)\, C_{p-2}\frac{D}{L\bar{V}_x} + p\left(\frac{2}{n+1} - \frac{n+3}{n+1}\xi^{n+1}\right) C_{p-1} \tag{5-7}$$

Equations (3) and (4) are also integrated to give

$$c_p(\xi, \phi, 0) = c_{p0}(\xi, \phi) \tag{5-8}$$

$$\frac{\partial C_p}{\partial \xi} = 0, \qquad \text{at} \quad \xi = 1 \qquad (5\text{-}9)$$

The average pth moment of the distribution can be obtained by summing Equation (7) over a cross section and then dividing by the cross-sectional area. The use of Green's theorem (28) and the boundary condition imposed by Equation (9) ensures that

$$\int_0^{2\pi} d\phi \int_0^1 \left[\frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial C_p}{\partial \xi} \right) + \frac{1}{\xi^2} \frac{\partial^2 C}{\partial \phi^2} \right] \xi \, d\xi = 0$$

because Green's theorem reduces the surface integral into a line integral along the wall of the tube and Equation (9) provides $\partial C_p/\partial \xi = 0$ at the tube wall. The average pth moment of the distribution thus takes the form

$$\frac{dM_p}{d\theta} = p(p-1)\, M_{p-2} \frac{D}{L\bar{V}_x} + \frac{p}{\pi} \int_0^{2\pi} d\phi \int_0^1 \xi \left(\frac{2}{n+1} - \frac{n+3}{n+1} \xi^{n+1} \right) C_{p-1} \, d\xi \qquad (5\text{-}10)$$

For p = 0, Equation (10) gives

$$\frac{dM_0}{d\theta} = 0 \qquad (5\text{-}11)$$

or m_0 = a constant, which may be taken as 1, since the total quantity of the tracer is constant. For p = 1, Equation (10) becomes

$$\frac{dM_1}{d\theta} = \frac{1}{\pi} \int_0^{2\pi} d\phi \int_0^1 \xi \left(\frac{2}{n+1} - \frac{n+3}{n+1} \xi^{n+1} \right) C_0 \, d\xi \qquad (5\text{-}12)$$

For p = 2, Equation (10) gives

$$\frac{dM_2}{d\theta} = \frac{2D}{L\bar{V}_x} + \frac{2}{\pi} \int_0^{2\pi} d\phi \int_0^1 \xi \left(\frac{2}{n+1} - \frac{n+3}{n+1} \xi^{n+1} \right) C_1 \, d\xi \qquad (5\text{-}13)$$

A general solution of C_0 and C_1 must be obtained in order that the first and second moments of the tracer distribution can be evaluated. Thus one needs to solve Equation (7) for $p = 0$ and $p = 1$. For $p = 0$, Equations (7) and (8) reduce to

$$\frac{\partial C_p}{\partial \theta} = \frac{LD}{R^2 \bar{V}_x} \left[\frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial C_p}{\partial \xi} \right) + \frac{1}{\xi^2} \frac{\partial^2 C_p}{\partial \phi^2} \right. \qquad (5\text{-}14)$$

and

$$C_0(\xi, \phi, 0) = C_{00}(\xi, \phi) \qquad (5\text{-}15)$$

The solution of Equation (14), subject to the boundary condition $\partial C_0 / \partial \xi = 0$ at $\xi = 1$, has been found to be (9, 20)

$$C_0(\xi, \phi, \theta) = \frac{1}{k} \left\{ 1 + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left[A_{mn}^0 \cos m\phi + B_{mn}^0 \sin m\phi \right] J_m(\alpha_{mn} \xi) \exp \left(-k\alpha_{mn}^2 \right) \right\} \qquad (5\text{-}16)$$

where $k = LD/R^2 \bar{V}_x$ and α_{mn} is the nth root of $dJ_m(\xi)/d\xi = 0$. The constants, A and B, can be determined by Equation (15) and making use of the orthogonality of the functions, that is (28)

$$\int_0^{2\pi} d\phi \int_0^1 \begin{matrix} A_{mn}^0 \cos^2 m\phi \\ B_{mn}^0 \sin^2 m\phi \end{matrix} \left\{ J_m(\alpha \xi) \right\}^2 \xi \, d\xi =$$

$$\int_0^{2\pi} d\phi \int_0^1 \begin{matrix} \cos m\phi \\ \sin m\phi \end{matrix} \; J_m(\alpha \xi) \cos (\xi, \phi) \, \xi \, d\xi \qquad (5\text{-}17)$$

Substituting Equation (16) into Equation (12) and integrating over the cross section, one obtains

$$\frac{dM_1}{d\theta} = \frac{1}{k} \sum_{n=1}^{\infty} A_{0n}^0 \, e^{-k\alpha^2 \theta} \int_0^1 2\xi \left(\frac{2}{n+1} - \frac{n+3}{n+1} \right) J_0(\alpha \xi) \, d\xi \qquad (5\text{-}18)$$

All the terms with $m \neq 0$ vanish by integration around the circle, because the integration of a sine function or a cosine function around a circle is zero. If n is a positive integer, the integration can easily be performed.

Integrating Equation (18) with respect to θ from 0 to θ, one obtains

$$M_1 = \frac{1}{k} \sum_{n=1}^{\infty} A_{0n}^{0} \alpha^{-2} \left[1 - \exp\,(-k\alpha^2\theta) \right] \int_0^1 2\xi \left(\frac{2}{n+1} - \frac{n+3}{n+1} \xi^{n+1} \right) J_0(\alpha\xi) d\xi \qquad (5\text{-}19)$$

As $\theta \rightarrow 0$, $M_{10} = 0$. This means that the center of gravity of the tracer moves with the mean speed of the flowing fluid if the solute concentration is initially uniform over a cross-sectional plane.

For $p = 1$, Equation (7) becomes

$$\frac{\partial C_1}{\partial \theta} = \frac{LD}{R^2 \bar{V}_x} \left[\frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial C_1}{\partial \xi} \right) + \frac{1}{\xi^2} \frac{\partial^2 C_1}{\partial \phi^2} \right] + \left(\frac{2}{n+1} - \frac{n+3}{n+1} \xi^{n+1} \right) C_0 \qquad (5\text{-}20)$$

in which C_0 is the same as given in Equation (16). The complete solution of C_1 consists of two parts, one is the complementary function which is (9, 20)

$$\frac{1}{k} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left[A_{mn}^{1} \cos m\phi + B_{mn}^{1} \sin m\phi \right] J_m(\alpha_{mn}\xi) \exp\,(-k\alpha_{mn}^2\theta)$$

The other is the particular solution which may be written as

$$\frac{1}{k} \cdot \left[-\frac{1}{2(n+1)} \xi^2 \left(1 - \frac{2}{n+3} \xi^{n+1} \right) + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left(A_{mn}^{0} \cos m\phi + B_{mn}^{0} \sin m\phi \right) \psi_{mn}(\xi) \exp\,(-k\alpha_{mn}^2\theta) \right]$$

Substituting this equation into Equation (20), it is found that $\psi_{mn}(\xi)$ must satisfy the equation

$$\frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \frac{\partial \psi}{\partial \xi} \right) + \left(\alpha^2 - \frac{m^2}{\xi^2} \right) \psi = - J_m(\alpha\xi) \left(\frac{2}{n+1} - \frac{n+3}{n+1} \xi^{n+1} \right)$$

When θ is large, the expression for C_1 reduces to

$$C_1 = \frac{1}{k}\left[-\frac{1}{2(n+1)}\xi^2(1-\xi^{n+1}) + 0\left\{\exp(-k\alpha^2\theta)\right\} + K\right] \qquad (5\text{-}21)$$

where $k = LD/R^2\bar{V}_x$ and K are constants. Substituting the value of C_1 into Equation (13) gives

$$\frac{dM_2}{d\theta} = \frac{2D}{L\bar{V}_x} - \frac{2R^2\bar{V}_x}{(n+1)LD}\int_0^1 \left(\frac{2}{n+1} - \frac{n+3}{n+1}\xi^{n+1}\right)\left(\xi^2 - \frac{2}{n+3}\xi^{n+3}\right)\xi\, d\xi$$

$$+ 0\left\{\exp(-k\alpha^2\theta)\right\}$$

$$= \frac{2D}{L\bar{V}_x} + \frac{R^2\bar{V}_x}{LD}\cdot\frac{1}{(n+3)(n+5)} \qquad (5\text{-}22)$$

Thus,

$$M_2 = \left[\frac{2D}{L\bar{V}_x} + \frac{R^2\bar{V}_x}{LD}\cdot\frac{1}{(n+2)(n+5)}\right]\theta + 0(\theta) + \text{a constant} \qquad (5\text{-}23)$$

and the constant is negligible by comparison.

For the axial-dispersed plug-flow model, Bischoff and Levenspiel obtained the following expressions for the moments:

$$M_0 = 1$$

$$M_1 = 0$$

$$M_2 = \frac{2\theta}{P_L} \qquad (5\text{-}24)$$

where $1/P_L$ is the dispersion coefficient for the axial-dispersed plug-flow model (17). Thus if V is the variance of the distribution of solute about the moving origin, it is reasonable to define the effective dispersion coefficient as

$$E = \lim_{t\to\infty}\frac{1}{2}\frac{dV}{dt} \qquad (5\text{-}25)$$

It should be noted that M is the dimensionless form of V. By comparing Equation (23) with Equation (24), E can be found to be

$$E = D + \frac{1}{2(n+3)(n+5)} \; \frac{R^2\bar{V}_x}{D} \qquad (5\text{-}26)$$

which is the sum of the molecular diffusion coefficient D and the apparent dispersion coefficient suggested by Taylor in his first paper (6). But there is no restriction on the value of their parameters.

For the case of fluid flow through a slit with the velocity V_x everywhere in the direction Ox and being a function of y (see Fig. (13)), the equation governing C is

$$\frac{\partial C}{\partial t} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right) - \left[\frac{1}{n+1} - \frac{n+2}{n+1}\left(\frac{y}{\delta}\right)^{n+1}\right]\frac{\partial C}{\partial x} \qquad (5\text{-}27)$$

The moving coordinates which move with the mean speed of flow are introduced in the equation above. Defining the dimensionless parameters

$$\eta = x/L\,, \qquad \zeta = |\,y/\delta\,|\,, \qquad \tau = z/\delta\,, \qquad \theta = t\bar{V}_x/L$$

the governing diffusion equation becomes

$$\frac{\partial C}{\partial \theta} = \frac{LD}{\delta^2\bar{V}_x}\left[\frac{\partial^2 C}{\partial \zeta^2} + \frac{\partial^2 C}{\partial \tau^2}\right] + \frac{D}{L\bar{V}_x}\frac{\partial^2 C}{\partial \eta^2} - \left(\frac{1}{n+1} - \frac{n+2}{n+1}\zeta^{n+1}\right)\frac{\partial C}{\partial \eta} \qquad (5\text{-}28)$$

The constant of integration is to be evaluated by using the initial and boundary conditions

$$C(\eta, \zeta, \tau, 0) = C_0(\eta, \zeta, \tau) \qquad (5\text{-}29)$$

$$\frac{\partial C}{\partial \zeta} = 0\,, \qquad \text{at } \zeta = 1 \qquad (5\text{-}30)$$

The pth moment of the distribution of solute in the filament at ζ, τ and at time θ and the pth moment of the solute in the tube are defined by

$$C_p(\zeta, \tau, \theta) = \int_{-\infty}^{\infty} \eta^p C(\eta, \zeta, \tau, \theta)\, d\eta \tag{5-31}$$

and

$$M_p(\theta) = \bar{C}_p = \int_0^1 d\tau \int_0^1 C_p(\zeta, \tau, \theta)\, d\zeta \tag{5-32}$$

Multiplying Equation (28) by η^p and integrating the resulting equation with respect to η from $-\infty$ to ∞ yield

$$\frac{\partial C_p}{\partial \theta} = \frac{LD}{\delta^2 \bar{V}_x}\left[\frac{\partial^2 C_p}{\partial \xi^2} + \frac{\partial^2 C_p}{\partial \tau^2}\right] + p(p-1) C_{p-2} \frac{D}{L\bar{V}_x} + pC_{p-1}\left(\frac{1}{n+1} - \frac{n+2}{n+1}\zeta^{n+1}\right) \tag{5-33}$$

The initial and boundary conditions become

$$C_p(\zeta, \tau, 0) = C_{p0}(\zeta, \tau) \tag{5-34}$$

$$\frac{\partial C_p}{\partial \zeta} = 0, \qquad \text{at} \quad \zeta = 1 \tag{5-35}$$

The pth moment of the solute in the slit is obtained by averaging Equation (33) over the cross section. The use of Green's theorem and Equation (35) insure that

$$\int_0^1 d\tau \int_0^1 \left[\frac{\partial^2 C_p}{\partial \xi^2} + \frac{\partial^2 C_p}{\partial \tau^2}\right] d\zeta = 0$$

Equation (33) is thus reduced to

$$\frac{dM_p}{d\theta} = p(p-1)\, M_{p-2} \frac{D}{L\bar{V}_x} + P\int_0^1 d\tau \int_0^1 \left(\frac{1}{n+1} - \frac{n+2}{n+1}\zeta^{n+1}\right) C_{p-1}\, d\zeta \tag{5-36}$$

For p = 0 Equation (36) becomes

$$\frac{dM_0}{d\theta} = 0 \qquad (5\text{-}37)$$

or $M_0 = 1$ since the total quantity of tracer may be taken as 1. For p = 1 and p = 2, Equation (36) gives

$$\frac{dM_1}{d\theta} = \int_0^1 d\tau \int_0^1 \left(\frac{1}{n+1} - \frac{n+2}{n+1}\zeta^{n+1}\right) C_0 \, d\zeta \qquad (5\text{-}38)$$

and

$$\frac{dM_2}{d\theta} = \frac{2D}{L\bar{V}_x} + \int_0^1 d\tau \int_0^1 \left(\frac{1}{n+1} - \frac{n+2}{n+1}\zeta^{n+1}\right) C_1 \, d\zeta \qquad (5\text{-}39)$$

which require general solutions of C_0 and C_1.

For p = 0 Equations (33) and (34) reduce to

$$\frac{\partial C_0}{\partial \theta} = \frac{LD}{\delta^2 \bar{V}_x}\left[\frac{\partial^2 C_0}{\partial \zeta^2} + \frac{\partial^2 C_0}{\partial \tau^2}\right] = k\left[\frac{\partial^2 C_0}{\partial \zeta^2} + \frac{\partial^2 C_0}{\partial \tau^2}\right] \qquad (5\text{-}40)$$

and

$$C_0(\zeta, \tau, 0) = C_{00}(\zeta, \tau) \qquad (5\text{-}41)$$

The function satisfying Equation (40) and given boundary conditions has been given as (9, 20)

$$C_0(\zeta, \tau, \theta) = \frac{1}{k}\left[1 + \sum_{n=1}^{\infty} A_n V_n(\zeta, \tau) \exp(-k\lambda_n \theta)\right] \qquad (5\text{-}42)$$

where λ_n is a sequence of eigenvalues and V_n is a complete set of orthogonal eigenfunctions satisfying the equation

$$\frac{\partial^2 V_n}{\partial \zeta^2} + \frac{\partial^2 V_n}{\partial \tau^2} + \lambda_n V_n = 0 \qquad (5\text{-}43)$$

and the boundary condition

$$\frac{\partial V_n}{\partial \zeta} = 0 \qquad \text{at} \qquad \zeta = 1 \qquad (5\text{-}44)$$

Inserting the value C_0 into Equation (38) and integrating the resulting equation with respect to θ from $\theta = 0$ to θ, it gives

$$M_1 = \frac{1}{k^2} \sum_{n=1}^{\infty} A_n \lambda_n \left[1 - \exp\,(-k\lambda_n \theta) \right] \int_0^1 d\tau \int_0^1 \left(\frac{1}{n+1} - \frac{n+2}{n+1} \zeta^{n+1} \right) V_n(\zeta,\, \tau)\, d\zeta \qquad (5\text{-}45)$$

For p = 2 Equation (33) is

$$\frac{\partial C_1}{\partial \theta} = \frac{LD}{\delta^2 \bar{V}_x} \left[\frac{\partial^2 C_1}{\partial \zeta^2} + \frac{\partial^2 C_1}{\partial \tau^2} \right] + \left(\frac{1}{n+1} - \frac{n+2}{n+1} \zeta^{n+1} \right) C_0 \qquad (5\text{-}46)$$

If we let $LD/\delta^2 \bar{V}_x = k$, the complementary solution of Equation (46) is

$$\frac{1}{k} \sum_{n=1}^{\infty} A_n^1 V_n(\zeta,\, \tau)\, \exp\,(-k\lambda_n \theta)$$

in which V_n is a set of eigenfunctions satisfying Equations (43) and (44). The particular solution may be written as

$$\frac{1}{k} \left[- \frac{\zeta^2}{2(n+1)} \left(1 - \frac{2}{n+3} \zeta^{n+1}\right) + \sum_{n=1}^{\infty} A_n \psi_n(\zeta,\, \tau)\, \exp\,(-k\lambda_n \theta) \right]$$

Substituting this equation into Equation (46), it is found that $\psi_n(\zeta,\, \tau)$ must satisfy the equation

$$\frac{\partial^2\psi_n}{\partial\zeta^2} + \frac{\partial^2\psi_n}{\partial\tau^2} + \lambda\psi_n = -\left(\frac{1}{n+1} - \frac{n+2}{n+1}\zeta^{n+1}\right) V_n(\zeta, \tau) \qquad (5\text{-}47)$$

When θ is large, the expression for C_1 may approximately be reduced to

$$C_1 = \frac{1}{k}\left[-\frac{\zeta^2}{2(n+1)}\left(1 - \frac{2}{n+3}\zeta^{n+1}\right) + O\left\{\exp\left(-k\lambda_n\theta\right)\right\} + K\right] \qquad (5\text{-}48)$$

where K is a constant. Inserting the value of C_1 into Equation (39), we have

$$\frac{dM_2}{d\theta} = \frac{2D}{L\bar{V}_x} + \int_0^1 d\tau \int_0^1 \left(\frac{1}{n+1} - \frac{n+2}{n+1}\zeta^{n+1}\right)\frac{1}{k}\left[-\frac{\zeta^2}{2(n+1)}\left(1 - \frac{2}{n+3}\zeta^{n+1}\right) + O\left\{\exp\left(-k\lambda_n\theta\right)\right\}\right]d\zeta$$

$$= \frac{2D}{L\bar{V}_x} + \frac{\delta^2\bar{V}_x}{LD}\cdot\frac{4}{3(n+4)(2n+5)}$$

Therefore,

$$M_2 = \left[\frac{2D}{L\bar{V}_x} + \frac{\delta^2\bar{V}_x}{LD}\cdot\frac{4}{3(n+4)(2n+5)}\right]\theta + \text{a constant}$$

Comparing this with Equation (25), the effective dispersion coefficient can be seen as

$$E = D + \frac{2}{3(n+4)(2n+5)}\;\frac{\delta^2\bar{V}_x}{D}$$

Aris (9) has shown that the dispersion model with its smaller number of parameters approaches normality for large time. Thus it is sufficient to describe the solute distributions by considering the first two moments, though the approach to normality for larger numbers of effective dispersion coefficients is slow.

VI. OUTLINE OF PROPOSED FUTURE WORK

The treatment of solute dispersion in the previous chapters is confined mainly to simple and generally useful two-constant and three-constant non-Newtonian models. Constant molecular diffusion coefficients, steady and isothermal flow situations and constant vessel geometries are assumed in order to simplify the analysis. On arriving at the end of the thesis, the writer is in a position to outline some of the areas which are connected with this work. Certainly, the methods used can be extended to more general problems.

Mathematical modeling is a useful and time saving tool, but its validity must be verified by experiment since some assumptions have been involved in the derivation. Radioactive isotopes, electrolytes or dye stuffs may be used as tracers. The following factors should be considered carefully in performing the experiment: roughness of pipe, bends, elbows and valves in the pipe, differences in physical properties between the two mixing fluids, and skewness of the observed profile in the analysis of experimental data.

In the derivation of the dispersion models, only the case without solute diffusion across the tube wall has been considered. Of course, this is not the only case worthy of consideration. Consider two phases flowing in co-axial annular regions with velocities and diffusion coefficient varying only with radial distance from the common axis. The inner and outer fluids may be referred to as the gas and liquid phase respectively, for which the resulting distribution of concentration can be used to interpret certain gas adsorption and chromatography systems.

Most investigations of fluid dispersion have been confined to simple vessel geometries. But many interesting problems arising in practical operations are those with non-Newtonian fluids in some specified conduits, such as flow of power-law model fluids in elliptic tubes and fluids flowing between cylinders which are not co-axial.

It has been pointed out (29) that Taylor and Aris discussed only the special case of injecting the tracer into a flow system, Aris' more rigorous study being restricted to a slug of material with initial finite moments with respect to the axial coordinate. It is anticipated that simpler and more convenient formulations can be developed from the properties of the exact solutions of the diffusion equation. It is obvious that a full solution of the diffusion equation would be very difficult. What we desire is an understanding of the gross features of dispersion. The variation of C(x, t) frequently provides the information desired. Thus, the formulation in terms of apparent Fick's diffusion down a gradient of C relative to the main motion is quite convenient for interpreting systems.

Much interest has been given to the flow of blood in the human cardiovascular system (30, 31, 32). Blood is apparently non-Newtonian even though it has been considered to be Newtonian in most investigations. A study of the blood flow system may be performed by considering the dispersion of non-Newtonian fluids associated with pulsating flow in tubes in which the walls have appreciable flexibility. Hapner (33) has studied such a system by means of dimensional analysis. The change of the pressure pulse resulting from changes in the dimensionless parameters has been studied.

A chemical engineer specifying mixing equipment needs to predict the flow pattern, average residence-time and residence-time distribution in case of continuous operations. A discussion of the literature prior to 1959 is presented by Bernhardt (34) and McKelvey(35). Significant progress has been made in the analysis of laminar mixing processes by the concept of shear deformation. Such theoretical analyses are described by Spencer and Wiley (36), Mohr, et al.(37) and Schrenk, et al.(38). An annular channel(38) and a tube (39) which continuously mix two polymer streams by rotation of the boundaries to produce the shear deformation are considered. Good agreement was reported between theory and experiment for the mixing pattern. For this type of mixing, performance data are needed for the various types of mixers available for mixing or blending Newtonian and non-Newtonian fluids of various viscosity ratios under laminar and turbulent flow regimes for various flow ratios of components.

ACKNOWLEDGEMENT

The author wishes to express his sincere appreciation to Dr. Liang-tseng Fan for his guidance and advice throughout the course of this work. His thanks are also extended to the Kansas State University Engineering Experimental Station for supporting this work, and the Computer Center of Kansas State University for use of the computing facilities. The author further extends his appreciation to Dr. W. H. Honstead and Dr. B. G. Kyle for reading the manuscript. The work was also partially supported by NSF Grant GK-67.

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NOMENCLATURE

C	concentration
C_0	concentration of tracer in inlet stream
C_m	area mean concentration
C_p	pth moment of the distribution of solute in the filament through ξ, ϕ at time θ
$\bar{C}_p$	pth moment of the distribution of solute in the tube
D	molecular diffusivity, l^2/t
d	inside diameter of round tube, l
E	effective or apparent dispersion coefficient, l^2/t
$\underline{E}(\theta)$	exit age distribution function
$\underline{F}(\theta)$	response curve to a step function input of tracer
f	Fanning friction factor
K	effective dispersion coefficient factor
k	mixing length constant
k	dimensionless parameter $LD/R^2\bar{V}_x$
L	characteristic length of system, l
M_p	pth moment of the distribution of solute in the tube
m	parameter in Ostwald-de Waele model
n	flow-behavior index in Ostwald-de Waele model
P	$p - \rho gz$
P_L	dispersion coefficient for the axial-dispersed plug-flow model
p	pressure, m/lt^2
Q	volumetric flow rate, l^3/t
R	tube radius, l
r	radial distance from center of tube, l

r_0	$\tau_0/(P_0 - P_L)/2L$
t	time, t
$\bar{t}$	mean residence-time, t
U_s	a unit step function
V_m	center line velocity, l/t
V_R	slip velocity at tube wall, l/t
V_x	axial velocity, l/t
$\bar{V}_x$	mean velocity in axial direction, l/t
V_*	friction velocity defined by $(\tau_0/\rho)^{\frac{1}{2}}$
x	axial distance, l
x_1	axial position relative to a coordinate system moving with the mean speed of flow, l
y	cross-sectional distance in rectangular coordinates, l
y_0	$\tau_0/(P_0 - P_L)L$

Greek Letters

α_n	n-th eigen-value
α	dimensionless constant
β	dimensionless constant
γ	wall slip velocity factor
$\overline{\overline{\Delta}}$	rate of deformation tensor
δ	characteristic length in the cross-sectional direction in rectangular coordinates, l
$\delta(\theta)$	Dirac delta function input
ϵ	coefficient of transfer
ζ	dimensionless position variable defined by y/δ
ζ_0	dimensionless flow behavior index in Bingham plastic model

η	dimensionless axial distance, x/L
η_1	dimensionless axial distance relative to a coordinate system moving with the mean speed of flow
θ	dimensionless time, $t/\bar{t}$ or $t\bar{V}_x/L$
λ	dimensionless constant
μ	mean of the r. t. d. f. or the first moment about the origin
μ	parameter in the Bingham model, m/lt
ν	parameter in the Ostwald-de Waele model
ξ	dimensionless radial distance, r/R
ξ_0	dimensionless flow behavior index in Bingham plastic model
ρ	fluid density, m/l^3
σ^2	variance of the r. t. d. f.
τ	dimensionless distance, z/δ
$\overset{\equiv}{\tau}$	shear stress tensor, m/t^2l
τ_0	parameter in Bingham plastic model, m/t^2l
φ	angle between the vertical direction and the x-coordinate
φ_0	parameter in Ellis model
φ_1	parameter in Ellis model

DISPERSION OF NON-NEWTONIAN FLUIDS

by

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B. S., National Taiwan University, 1962

AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

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1965

A study has been made of the dispersion of non-Newtonian fluids in continuous flow systems. If a pulse of tracer is injected into a flowing stream, this discontinuity spreads out as it moves with the fluid past a downstream measurement point. The amount of spreading depends on the intensity of dispersion in the system. It has been proved that the variance of the tracer distribution and the effective dispersion coefficient can be easily related by means of Taylor's analysis.

Dispersion by convection alone, which is usually called the convective model, is considered first. Discontinuous points do occur in the resulting distribution curves. The deformation in the shape of the actual residence-time distribution curve is caused by the dispersion effect.

The dispersion model assumes that the combined effects of the variation of axial velocity with cross-sectional position and the cross-sectional material transport by molecular diffusion are the dominant factors causing the dispersion. Conditions under which such assumptions are valid are investigated. Expressions for the residence-time distributions as a correlation of the dispersion coefficients are obtained.

An alternate approach is that which characterizes the dispersion in terms of its moments. These moments are related simply to measurable quantities. This approach is used to interpret the dispersion of the Ostwald-de Waele fluid.